

# The universality principle for spectral distributions of sample covariance matrices.

Pavel Yaskov<sup>1</sup>

## Abstract

We derive the universality principle for empirical spectral distributions of sample covariance matrices and their Stieltjes transforms. This principle states the following. Suppose quadratic forms of random vectors  $\mathbf{y}_p$  in  $\mathbb{R}^p$  satisfy a weak law of large numbers and the sample size grows at the same rate as  $p$ . Then the limiting spectral distribution of corresponding sample covariance matrices is the same as in the case with conditionally Gaussian  $\mathbf{y}_p$ . This result is generalized for  $m$ -dependent martingale difference sequences and  $m$ -dependent linear processes.

**Keywords:** Random matrices; Universality; Sample covariance matrix.

## 1 Introduction

The random matrix theory plays an important role in modern high-dimensional statistics (e.g., see [16]). A large-dimensional sample covariance matrix is an object of primary interest. Many test statistics could be defined by its eigenvalues (e.g., see [2] and [12]). Asymptotic behaviour of such statistics depends on the empirical distribution of the eigenvalues. The latter is called the empirical spectral distribution (ESD).

---

<sup>1</sup>Steklov Mathematical Institute, Russia  
e-mail: yaskov@mi.ras.ru

Supported by RNF grant 14-21-00162 from the Russian Scientific Fund.

The universality principle for ESDs of sample covariance matrices says that the limiting behaviour of ESDs is the same as when a random sample is taken from a Gaussian distribution. In the pioneering paper [13], Marchenko and Pastur discovered general conditions implying universality. Namely, if quadratic forms of random vectors  $\mathbf{y}_p$  in  $\mathbb{R}^p$  concentrate near their expectations, then ESDs of corresponding sample covariance matrices obeys the universality principle. Bai and Zhou [2] gave the first formal proof of this fact (see also the paper of Pajor and Pastur [18] and the book of Pastur and Shcherbina [19]). They assumed that entries of  $\mathbf{y}_p$  have finite fourth moments. Girko and Gupta [7] considered the universality principle without the finite fourth-moment assumption (but under a more restrictive assumption on covariance matrices). All of these results are particular cases of a general universality principle that is derived in the present paper.

Recall that a sample covariance matrix in the random matrix theory is usually defined by  $n^{-1}\mathbf{Y}_{pn}\mathbf{Y}_{pn}^\top$ , where  $\mathbf{Y}_{pn}$  is a  $p \times n$  random matrix whose columns are independent copies of  $\mathbf{y}_p$ . The average of these columns is not subtracted since it does not affect the limiting spectral distribution (see Chapter 3 in [1]).

The present paper contributes to the random matrix theory in two directions. First, it shows when one has the universality principle for ESDs in a very general framework. Namely, if a weak law of large numbers for quadratic forms of  $\mathbf{y}_p$  holds, then the limiting spectral distribution of  $n^{-1}\mathbf{Y}_{pn}\mathbf{Y}_{pn}^\top$  is the same as in the case of conditionally Gaussian  $\mathbf{y}_p$  when  $n$  grows at the same rate as  $p$ . Similar conditions have appeared in the literature in a much stronger form. E.g., see Theorem 1.1 in [2], Theorem 19.1.8 in [19] and Theorem 6.1 in [7].

We generalize these results for random matrices  $\mathbf{Y}_{pn}$  whose columns form an  $m$ -dependent martingale difference sequence or an  $m$ -dependent linear process. Recently Banna, Merlevede and Peligrad [5] obtained the universality principle assuming  $m$ -

dependence in rows and columns of  $\mathbf{Y}_{pn}$ . However, the technique developed in [5] allows to derive this property only in the case where the limiting spectral distribution of  $n^{-1}\mathbf{Y}_{pn}\mathbf{Y}_{pn}^\top$  is completely determined by the covariance structure of  $\mathbf{Y}_{pn}$ 's entries. However, in general, it is not determined.

Second, we derive useful moment inequalities for quadratic forms. These inequalities show when a weak law of large numbers holds. The latter allows us to describe explicitly a wide class of  $\mathbf{y}_p$  which sample covariance matrices have universality properties.

The paper is structured as follows. Sections 2 and 3 contain universality laws for ESDs and their Stieltjes transforms. Section 4 deals with moment inequalities for quadratic forms. The proofs are given in Section 5 and Appendix.

## 2 Universality of ESD: independent observations.

For each  $p \geq 1$ , let  $\mathbf{y}_p$  be a random vector in  $\mathbb{R}^p$  and  $\Sigma_p$  be a random symmetric positive semi-definite  $p \times p$  matrix defined on the same probability space. Consider the following assumptions.

**(A1)**  $(\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p))/p \xrightarrow{P} 0$  as  $p \rightarrow \infty$  for all sequences of real symmetric positive semi-definite  $p \times p$  matrices  $A_p$  with uniformly bounded spectral norms  $\|A_p\|$ .

**(A2)**  $\text{tr}(\Sigma_p^2)/p^2 \xrightarrow{P} 0$  as  $p \rightarrow \infty$ .

Assumption (A1) is a version of the weak law of large numbers for quadratic forms. It is a key assumption. Obviously, it is satisfied for  $\Sigma_p = \mathbf{y}_p \mathbf{y}_p^\top$ . However, (A2) may not hold in this case. Assumption (A2) guarantees that (A1) holds when, conditionally on  $\Sigma_p$ ,  $\mathbf{y}_p$  has a centred normal distribution. This is shown in the next proposition (for a proof, see Appendix).

**Proposition 2.1.** *Let  $\mathbf{y}_p = \Sigma_p^{1/2} \mathbf{w}_p$ , where  $\Sigma_p^{1/2}$  is the principal square root of  $\Sigma_p$  and  $\mathbf{w}_p$  is a standard normal vector in  $\mathbb{R}^p$  that is independent of  $\Sigma_p$  for each  $p \geq 1$ . Then (A1) holds if and only if (A2) holds. Moreover,*

$$\mathbb{P}(|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)| > \varepsilon p) \leq \mathbb{E} \min \left\{ \frac{16M^2 \text{tr}(\Sigma_p^2)}{(\varepsilon p)^2}, 1 \right\}$$

for any  $\varepsilon, M > 0$  and each complex  $p \times p$  matrix  $A_p$  with spectral norm  $\|A_p\| \leq M$ .

Let  $\mathbf{Y}_{pn}$  be a  $p \times n$  matrix which columns are independent copies of  $\mathbf{y}_p$  and  $\mathbf{Z}_{pn}$  be a  $p \times n$  matrix which columns are independent copies of  $\mathbf{z}_p = \Sigma_p^{1/2} \mathbf{w}_p$ , where  $\mathbf{w}_p$  is given in Proposition 2.1. The universality principle for ESDs states that ESD of  $n^{-1} \mathbf{Y}_{pn} \mathbf{Y}_{pn}^\top$  asymptotically behaves in the same manner as ESD of  $n^{-1} \mathbf{Z}_{pn} \mathbf{Z}_{pn}^\top$ . Recall that ESD of a  $p \times p$  symmetric matrix  $A$  is uniquely defined by its Stieltjes transform in the upper-half plane

$$s(z) = p^{-1} \text{tr}(A - zI_p)^{-1}, \quad z \in \mathbb{C}^+ = \{w \in \mathbb{C} : \text{Im}(w) > 0\},$$

where  $I_p$  is the  $p \times p$  identity matrix. The following theorem deals with universality properties of Stieltjes transforms of  $n^{-1} \mathbf{Y}_{pn} \mathbf{Y}_{pn}^\top$ .

**Theorem 2.2.** *If (A1) and (A2) hold, then*

$$p^{-1} \text{tr}(n^{-1} \mathbf{Y}_{pn} \mathbf{Y}_{pn}^\top - zI_p)^{-1} - p^{-1} \text{tr}(n^{-1} \mathbf{Z}_{pn} \mathbf{Z}_{pn}^\top - zI_p)^{-1} \rightarrow 0 \quad a.s. \quad (1)$$

for all  $z \in \mathbb{C}^+$  as  $n \rightarrow \infty$ , where  $p = p(n)$  is such that  $p/n \rightarrow y$  for some  $y > 0$ .

**Remark 1.** Theorem 2.2 can be extended to the case where columns of  $\mathbf{Y}_{pn}$  are not identically distributed. Namely, let  $(\mathbf{y}_{pk}, \Sigma_{pk})$  be independent over  $k = 1, \dots, n$ ,  $\mathbf{Y}_{pn}$  and  $\mathbf{Z}_{pn}$  be  $p \times n$  matrices with columns  $\{\mathbf{y}_{pk}\}_{k=1}^n$  and  $\{\Sigma_{pk}^{1/2} \mathbf{w}_{pk}\}_{k=1}^n$  correspondingly,

where  $\{\mathbf{w}_{pk}\}_{k=1}^n$  are independent standard normal vectors that are also independent of  $\{\Sigma_{pk}\}_{k=1}^n$ . The conclusion of Theorem 2.2 holds when the following assumptions hold for given  $p = p(n)$  with  $p(n) \rightarrow \infty$  as  $n \rightarrow \infty$ .

**(A3)** For any  $M, \varepsilon > 0$ ,

$$\frac{1}{n} \sum_{k=1}^n \sup_{A_p} \mathbb{P}(|\mathbf{y}_{pk}^\top A_p \mathbf{y}_{pk} - \text{tr}(\Sigma_{pk} A_p)| > \varepsilon p) \rightarrow 0, \quad n \rightarrow \infty,$$

where each supremum is taken over all real symmetric positive semi-definite  $p \times p$  matrices  $A_p$  with spectral norms  $\|A_p\| \leq M$ .

**(A4)** For any  $\varepsilon > 0$ ,

$$\frac{1}{n} \sum_{k=1}^n \mathbb{P}(\text{tr}(\Sigma_{pk}^2) > \varepsilon p^2) \rightarrow 0, \quad n \rightarrow \infty.$$

In practice, Theorem 2.2 is supposed to be used with the next well-known proposition (e.g., see Exercise 2.4.10 in [20] for a more general statement).

**Proposition 2.3.** *Let  $p = p(n) \rightarrow \infty$  as  $n \rightarrow \infty$ , and  $A_n$  is a real random symmetric  $p \times p$  matrix for each  $n \geq 1$ . If, for each  $z \in \mathbb{C}^+$ ,*

$$p^{-1} \text{tr}(A_n - zI_p)^{-1} \rightarrow s(z) \quad a.s.$$

*as  $n \rightarrow \infty$  for a deterministic function  $s = s(z)$ , then*

$$s(z) = \int_{-\infty}^{\infty} \frac{F(d\lambda)}{z - \lambda}$$

*is the Stieltjes transform of some distribution function  $F = F(\lambda)$  on  $\mathbb{R}$  and*

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} F_n(\lambda) = F(\lambda) \text{ for any continuity point } \lambda \text{ of } F\right) = 1,$$

where  $F_n(\lambda) = \sum_{k=1}^p I(\lambda_{kn} \leq \lambda)/p$ ,  $\lambda \in \mathbb{R}$ , for the set  $\{\lambda_{kn}\}_{k=1}^p$  of eigenvalues of  $A_n$ .

Due to Proposition 2.3, to prove that ESD of  $n^{-1}\mathbf{Y}_{pn}\mathbf{Y}_{pn}^\top$  converges vaguely, one should only check that Stieltjes transforms of  $n^{-1}\mathbf{Z}_{pn}\mathbf{Z}_{pn}^\top$  from Theorem 2.2 converge.

Let us now consider three important particular cases.

**Case 1.**  $\Sigma_p$ ,  $p \geq 1$ , are **deterministic matrices**. In this case,

$$\mathbf{Z}_{pn}\mathbf{Z}_{pn}^\top = \Sigma_p^{1/2}\mathbf{W}_{pn}\mathbf{W}_{pn}^\top\Sigma_p^{1/2},$$

where  $\mathbf{W}_{pn}$  is a  $p \times n$  random matrix with independent standard normal entries. The set of eigenvalues of

$$\Sigma_p^{1/2}\mathbf{W}_{pn}\mathbf{W}_{pn}^\top\Sigma_p^{1/2}$$

is the same as that of  $\mathbf{W}_{pn}^\top\Sigma_p\mathbf{W}_{pn}$  (when excluding  $|p - n|$  zero eigenvalues). In addition, since the distribution  $\mathbf{W}_{pn}$  does not change when multiplying  $\mathbf{W}_{pn}$  by an orthogonal matrix,

$$\mathbf{W}_{pn}^\top\Sigma_p\mathbf{W}_{pn} \stackrel{d}{=} \mathbf{W}_{pn}^\top D_p \mathbf{W}_{pn},$$

where  $\stackrel{d}{=}$  is the equality in law and  $D_p$  is a diagonal matrix whose diagonal entries are eigenvalues of  $\Sigma_p$ . Hence, the limiting spectral distribution of  $n^{-1}\mathbf{W}_{pn}^\top D_p \mathbf{W}_{pn}$  could be derived from Theorem 4.3 in [1] (or Theorem 7.2.2 in [19]), when ESD of  $D_p$  converges weakly to a probability measure on  $\mathbb{R}_+$ .

A very important case is  $\Sigma_p = D_p = I_p$ . In order to give some precise statements, we recall that the Marchenko-Pastur law with parameter  $y > 0$  has a distribution function

$$F_y(\lambda) = \max\{1 - 1/y, 0\}I(\lambda \geq 0) + I(\lambda \in [a, b]) \int_a^\lambda \frac{\sqrt{(b-x)(x-a)}}{2\pi xy} dx,$$

where  $\lambda \in \mathbb{R}$ ,  $a = (1 - \sqrt{y})^2$  and  $b = (1 + \sqrt{y})^2$ .

**Theorem 2.4.** *Let (A1) holds for  $\Sigma_p = I_p$  and  $p = p(n)$ . Then, with probability one, ESD of  $\mathbf{Y}_{pn}\mathbf{Y}_{pn}^\top/n$  converges vaguely to the Marchenko-Pastur law with parameter  $y > 0$  as  $n \rightarrow \infty$  and  $p/n \rightarrow y$ .*

**Case 2.**  $\Sigma_p = \xi I_p$ ,  $p \geq 1$ , for a random variable  $\xi$ . In this case,

$$\mathbf{Z}_{pn}\mathbf{Z}_{pn}^\top = \mathbf{W}_{pn}T_n\mathbf{W}_{pn}^\top,$$

where  $\mathbf{W}_{pn}$  is as above and  $T_n$  is a diagonal matrix with diagonal entries that are independent copies of  $\xi^2$ . By the Glivenko-Cantelli theorem, ESD of  $T_n$  converges weakly a.s. to the distribution of  $\xi^2$ . Hence, the limiting spectral distribution of  $n^{-1}\mathbf{W}_{pn}^\top T_n \mathbf{W}_{pn}$  can be derived from Theorem 4.3 in [1] (or Theorem 7.2.2 in [19]).

**Case 3.**  $\Sigma_p$ ,  $p \geq 1$ , are random diagonal matrices. In this case, conditionally on  $\{\Sigma_{pk}\}_{k=1}^n$ , the matrix  $\mathbf{Z}_{pn}$  consists of centred independent normal variables with different variances in general. There are some particular results that allow to calculate the limiting spectral distribution in this case. E.g., see Theorem 2 in [8] (or Theorem 1 in [3]). However, we are not aware of any general result.

To apply Theorem 2.2, we need tools to verify (A1). This could be done via moment inequalities for quadratic forms given in Section 4.

### 3 Universality of ESD: $m$ -dependent observations.

In this section, we generalize results from Section 1 to the  $m$ -dependent case.

For each  $p, n \geq 1$  and  $k = 1, \dots, n$ , let  $y_{pk}$  be a random vector in  $\mathbb{R}^p$  and  $\Sigma_{pk}$  be

a symmetric random positive semi-definite  $p \times p$  matrix. Define

$$\mathcal{F}_k^p = \sigma((y_{pj}, \Sigma_{pj}), 1 \leq j \leq k), \quad 1 \leq k \leq n,$$

$\mathcal{F}_0^p$  to be the trivial  $\sigma$ -algebra and  $\mathbb{E}_k = \mathbb{E}(\cdot | \mathcal{F}_k^p)$  for  $k = 0, 1, \dots, n$ .

Assume that  $\{(\mathbf{y}_{pk}, \Sigma_{pk})\}_{k=1}^n$  is an  $m$ -dependent sequence for some  $m = m(n)$  and  $p = p(n)$ , i.e.  $\sigma$ -algebras  $\mathcal{F}_k^p$  and  $\sigma((\mathbf{y}_{pj}, \Sigma_{pj}), k+m \leq j \leq n)$  are independent for each  $k = 1, \dots, n-m$ . Introduce the following assumption.

**(A5)** For given  $p = p(n)$  and  $m = m(n)$ , as  $n \rightarrow \infty$ ,

$$\sum \mathbb{E}(\|\mathbb{E}_k \Sigma_{pl}\| + \|\mathbb{E}_k \mathbf{y}_{pl} \mathbf{y}_{pl}^\top\|) \text{tr}(\mathbf{y}_{pk} \mathbf{y}_{pk}^\top + \Sigma_{pk}) = o(n^3),$$

where the sum is taken over all  $(k, l)$  with  $1 \leq k < l \leq n$  and  $l < k+m$ . Suppose also

$$\sum (\|\mathbb{E} \Sigma_{pl}\| + \|\mathbb{E} \mathbf{y}_{pl} \mathbf{y}_{pl}^\top\|) \text{tr}(\mathbb{E} \mathbf{y}_{pk} \mathbf{y}_{pk}^\top + \mathbb{E} \Sigma_{pk}) = o(n^3),$$

where the sum is taken over all  $(k, l)$  with  $1 \leq k, l \leq n$  and  $m \leq |k-l| < 2m$ .

**Theorem 3.1.** *Let (A3), (A4) and (A5) hold. If  $m = m(n)$  and  $p = p(n)$  satisfy  $m = o(n/\log n)$  and  $p/n \rightarrow y > 0$  for some  $y > 0$  as  $n \rightarrow \infty$ , then*

$$p^{-1} \text{tr}(n^{-1} \mathbf{Y}_{pn} \mathbf{Y}_{pn}^\top - z I_p)^{-1} - p^{-1} \text{tr}(n^{-1} \mathbf{Z}_{pn} \mathbf{Z}_{pn}^\top - z I_p)^{-1} \rightarrow 0 \quad a.s.$$

for all  $z \in \mathbb{C}^+$  as  $n \rightarrow \infty$ , where  $\mathbf{Z}_{pn}$  and  $\mathbf{Y}_{pn}$  are defined in Remark 1.

We now discuss when all assumption of Theorem 3.1 hold. The key assumptions are (A3) and (A5). Assumption (A3) could be verified via moment inequalities for quadratic forms given in Section 4. Simple conditions implying (A5) are given in the next proposition.



**Proposition 3.2.** *Let*

$$\max_{1 \leq k \leq n} (\|\mathbb{E}_{k-1}(\Sigma_{pk})\| + \|\mathbb{E}_{k-1}(\mathbf{y}_{pk}\mathbf{y}_{pk}^\top)\|) = o(n/m) \quad a.s.$$

for some given  $p = p(n)$  and  $m = m(n)$ . If, in addition,

$$\sum_{k=1}^n \text{tr}(\mathbb{E}\mathbf{y}_{pk}\mathbf{y}_{pk}^\top + \mathbb{E}\Sigma_{pk}) = O(n^2),$$

then (A5) hold.

**Remark 2.** Assumption (A5) implicitly restricts  $\{\mathbf{y}_{pk}\}_{k=1}^n$  to be an *almost* martingale difference sequence when  $p = p(n)$  grows at the same rate as  $n$ . By *almost*, we mean that most entries of  $\mathbb{E}_{k-1}\mathbf{y}_{pk}$  are close to zero for most  $k = 1, \dots, n$ . Alternatively, one may say that  $\|\mathbb{E}_{k-1}\mathbf{y}_{pk}\|^2 = o(n)$  with large probability for most  $k$ . To see this, consider the following setting. Let  $\mathbb{E}\mathbf{y}_{pk}\mathbf{y}_{pk}^\top = I_p$  and  $\|\mathbb{E}_{k-1}\mathbf{y}_{pk}\|^2 \geq Cn$  a.s. for some  $C > 0$  and all  $1 \leq k \leq n$  ( $n \geq 1$ ), then

$$\|\mathbb{E}_{k-1}\mathbf{y}_{pk}\mathbf{y}_{pk}^\top\| \geq \|\mathbb{E}_{k-1}\mathbf{y}_{pk}(\mathbb{E}_{k-1}\mathbf{y}_{pk})^\top\| = \|\mathbb{E}_{k-1}\mathbf{y}_{pk}\|^2 \geq Cn, \quad 2 \leq k \leq n,$$

and

$$\sum_{k=2}^n \mathbb{E}\|\mathbb{E}_{k-1}\mathbf{y}_{pk}\mathbf{y}_{pk}^\top\| \text{tr}(\mathbf{y}_{p(k-1)}\mathbf{y}_{p(k-1)}^\top) \geq Cn^3.$$

Clearly (A5) doesn't hold in this case.

**Example 1.** Suppose  $m \geq 1$  is fixed and  $p = p(n)$ . Consider independent identically distributed  $\mathbb{R}^p$ -valued random vectors  $\mathbf{e}_{pk}$ ,  $-m < k \leq n$ , which entries are independent copies of a random variable  $e$  with  $\mathbb{E}e = 0$  and  $\mathbb{E}e^2 = 1$ . Let  $\mathbf{y}_{pk} = A_{pk}\mathbf{e}_{pk}$  for all  $1 \leq k \leq n$ , where  $A_{pk}$  is a  $p \times p$  random matrix measurable w.r.t.  $\sigma(\mathbf{e}_{pl}, k-m < l < k)$ .

Let also

$$\frac{1}{n} \sum_{k=1}^n \frac{\text{tr}(\mathbb{E}\Sigma_{pk}^2)}{p^2} = o(1) \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^n \frac{\text{tr}(\mathbb{E}\Sigma_{pk})}{p} = O(1), \quad n \rightarrow \infty,$$

where  $\Sigma_{pk} = A_{pk}A_{pk}^\top$ ,  $1 \leq k \leq n$ . The latter yields (A3) and (A4) (see Corollary 4.9 below). Note also that  $\mathbb{E}\Sigma_{pk} = \mathbb{E}\mathbf{y}_{pk}\mathbf{y}_{pk}^\top$ ,  $1 \leq k \leq n$ . If, in addition,

$$\max_{1 \leq k \leq n} \|\Sigma_{pk}\| = o(n) \quad \text{a.s.},$$

then (A5) hold (for  $p = O(n)$ ). This can be proven similarly to Proposition 3.2.

We now extend previous results to the case of linear  $m$ -independent processes. As in Section 2, for each  $p \geq 1$ , let  $\mathbf{y}_p$  be a random vector in  $\mathbb{R}^p$  and  $\Sigma_p$  be a random symmetric positive semi-definite  $p \times p$  matrix defined on the same probability space.

Let  $\mathbf{Y}_{pn}$  be a  $p \times n$  matrix which columns are independent copies of  $\mathbf{y}_p$  and  $\mathbf{Z}_{pn}$  be a  $p \times n$  matrix which columns are independent copies of  $\mathbf{z}_p = \Sigma_p^{1/2} \mathbf{w}_p$ , where  $\mathbf{w}_p$  is a standard normal vector independent of  $\Sigma_p$ . Suppose  $L_n = (l_{kj})_{k,j=1}^n$  is a lower triangular  $m$ -banded  $n \times n$  matrix for each  $n \geq 1$  with entries  $l_{kj} = l_{kj}(n)$  that are equal to zero when  $j \leq k - m$  or  $j > k$ . Then

$$\frac{1}{n} \mathbf{Y}_{pn} L_n^\top L_n \mathbf{Y}_{pn}^\top = \frac{1}{n} \sum_{k=1}^n \left( \sum_{j: k-m < j \leq k} l_{kj} \mathbf{y}_{pj} \right) \left( \sum_{j: k-m < j \leq k} l_{kj} \mathbf{y}_{pj} \right)^\top$$

with  $j \in \{1, \dots, n\}$ . Clearly, the sequence

$$\left\{ \sum_{j: k-m < j \leq k} l_{kj} \mathbf{y}_{pj} \right\}_{k=1}^n$$

is an  $m$ -dependent linear process. To state the universality principle for these sequences, we need one more assumption.

**(A6)**  $(\|\mathbb{E}\mathbf{y}_p\mathbf{y}_p^\top\| + \|\mathbb{E}\Sigma_p\|)\text{tr}(\mathbb{E}\mathbf{y}_p\mathbf{y}_p^\top + \mathbb{E}\Sigma_p) = o(p^2)$  as  $p \rightarrow \infty$ .

**Theorem 3.3.** *If (A1), (A2) and (A6) hold,  $m$  is a fixed natural number, and entries of  $L_n$  are uniformly bounded over  $n$ , then*

$$p^{-1}\text{tr}(n^{-1}\mathbf{Y}_{pn}L_n^\top L_n\mathbf{Y}_{pn}^\top - zI_p)^{-1} - p^{-1}\text{tr}(n^{-1}\mathbf{Z}_{pn}L_n^\top L_n\mathbf{Z}_{pn}^\top - zI_p)^{-1} \rightarrow 0 \quad a.s. \quad (2)$$

for all  $z \in \mathbb{C}^+$  as  $n \rightarrow \infty$ , where  $p = p(n)$  is such that  $p/n \rightarrow y$  for some  $y > 0$ .

**Remark 3.** Theorem 3.3 can be extended to the case where columns of  $\mathbf{Y}_{pn}$  are not identically distributed similarly to Theorem 2.2 (see Remark 1). It can be also extended to the case where  $m = m(n)$  grows to infinity as  $n \rightarrow \infty$ .

## 4 Moment inequalities for quadratic forms

Let  $\{X_k\}_{k=1}^\infty$  be a sequence of random variables and  $\{\varphi_k\}_{k=1}^\infty$  be a sequence of non-negative numbers such that

$$|\mathbb{E}X_iX_jX_kX_l| \leq \min\{\varphi_{j-i}, \varphi_{k-j}, \varphi_{l-k}\} \quad \text{for all } i < j < k < l. \quad (3)$$

Set  $\mathbf{x}_p = (X_1, \dots, X_p)$  for any  $p \geq 1$ . Consider the following assumption.

**(B1)**  $\Phi_0 + \Phi_1 < \infty$ , where  $\Phi_0 = \sup\{\mathbb{E}X_k^4 : k \geq 1\}$  and  $\Phi_1 = \sum_{k=1}^\infty k\varphi_k$ .

**Theorem 4.1.** *If (B1) holds, then, for any  $a \in \mathbb{R}^p$  and all real  $p \times p$  matrices  $A$  with zero diagonal,*

$$\mathbb{E}|\mathbf{x}_p^\top a|^4 \leq C(\Phi_0 + \Phi_1)\|a\|^4 \quad \text{and} \quad \mathbb{E}|\mathbf{x}_p^\top A\mathbf{x}_p|^2 \leq C(\Phi_0 + \Phi_1)\text{tr}(AA^\top)$$

for some universal constant  $C > 0$ , where  $\|a\| = \sqrt{a^\top a}$ .

The proof of Theorem 4.1 is based on the strategy developed by Gaposhkin in [6].

**Corollary 4.2.** *If  $\mathbf{y}_p = \mathbf{x}_p$ ,  $p \geq 1$ . If (B1) holds, then (A1) and (A2) hold for diagonal matrices  $\Sigma_p$  with diagonal entries  $X_1^2, \dots, X_p^2$ .*

Let now  $\{\phi_k\}_{k=1}^\infty$  be a sequence of non-negative numbers such that

$$\text{Cov}(X_i^2, X_j^2) \leq \phi_{j-i} \quad \text{for all } i < j. \quad (4)$$

Introduce the following assumption.

**(B2)**  $\Phi_0 + \Phi_1 + \Phi_2 < \infty$ , where  $\Phi_0, \Phi_1$  are given in (B1) and  $\Phi_2 = \sum_{k=1}^\infty \phi_k$ .

**Theorem 4.3.** *If (B2) holds, then, for all real  $p \times p$  matrices  $A$ ,*

$$\mathbb{E}|\mathbf{x}_p^\top A \mathbf{x}_p - \text{tr}(\Sigma_p A)|^2 \leq C(\Phi_0 + \Phi_1 + \Phi_2) \text{tr}(AA^\top)$$

for a universal constant  $C > 0$ , where  $\Sigma_p$  is a diagonal matrix with diagonal entries  $\mathbb{E}X_1^2, \dots, \mathbb{E}X_p^2$ .

**Remark 4.** Lemma 2.5 in [18] and Lemma 19.1.4 in [19] contain some related estimates for isotropic  $\mathbf{x}_p$  with a log-concave distribution.

Since  $\text{tr}(A_p A_p^\top) \leq p \|A_p\|^2$  for all real  $p \times p$  matrices  $A_p$ , we have the following version of the law of large numbers under assumptions of Theorem 4.3.

**Corollary 4.4.** *Let  $\mathbf{y}_p = \mathbf{x}_p$ ,  $p \geq 1$ . If (B2) holds, then (A1) and (A2) hold for the diagonal matrix  $\Sigma_p$  with diagonal entries  $\mathbb{E}X_1^2, \dots, \mathbb{E}X_p^2$ .*

The above theorems assume that all  $X_k$  have finite fourth moments. It can be a restrictive assumption in some applications. Assuming a martingale-type dependence in  $X_k$ , one can relax this assumption.

**(B3)** Let  $\{X_k\}_{k=1}^\infty$  is a martingale difference sequence w.r.t. its own filtration and there is  $M > 0$  such that  $\mathbb{E}X_1^2 \leq M$  and  $\mathbb{E}[X_k^2|X_1, \dots, X_{k-1}] \leq M$  a.s. for all  $k \geq 2$ .

**Theorem 4.5.** *Let (B3) holds. Then, for all real  $p \times p$  matrices  $A$  with zero diagonal,*

$$\mathbb{E}|\mathbf{x}_p^\top A \mathbf{x}_p|^2 \leq 2M^2 \text{tr}(AA^\top). \quad (5)$$

Let us consider a stronger alternative to (B3).

**(B4)** Let  $\{X_k\}_{k=1}^\infty, \{X_k^2 - 1\}_{k=1}^\infty$  are martingale difference sequences w.r.t. their own filtrations.

**Theorem 4.6.** *If (B4) holds, then*

$$\mathbb{E}|\mathbf{x}_p^\top A \mathbf{x}_p - \text{tr}(A)| \leq Cb\sqrt{\text{tr}(AA^\top)} + C \sum_{k=1}^p |a_{kk}| \mathbb{E}|X_k^2 - 1| I(|X_k^2 - 1| > b^2) \quad (6)$$

for all real  $p \times p$  matrices  $A = (a_{ij})_{i,j=1}^p$ , each  $b > 1$  and a universal constant  $C > 0$ .

**Remark 5.** Using arguments similar to those in the proof of Theorem 4.6, one can derive similar inequalities for  $m$ -dependent sequences  $\{X_k\}_{k=1}^\infty$ . The universality principle for such sequences is proved in [11].

**Corollary 4.7.** *If  $\mathbf{y}_p = \mathbf{x}_p$ ,  $p \geq 1$ , and*

$$\frac{1}{p} \sum_{k=1}^p \mathbb{E}X_k^2 I(|X_k| > \varepsilon\sqrt{p}) \rightarrow 0 \quad \text{for all } \varepsilon > 0. \quad (7)$$

*If (B4) holds, then (A1) and (A2) hold for diagonal matrices  $\Sigma_p = I_p$ ,  $p \geq 1$ .*

Corollary 4.7 can be derived from Theorem 4.6 by taking  $b = \varepsilon\sqrt{p}$ , tending  $p \rightarrow \infty$  and then tending  $\varepsilon \rightarrow 0$ .

Let us now give examples of  $\{X_k\}_{k=1}^\infty$  that satisfy the above assumptions.

**Example 2.** If  $\{X_k\}_{k=1}^\infty$  are independent random variables with uniformly bounded forth moments and  $\mathbb{E}X_k = 0$ , then (3) and (4) hold for  $\varphi_k = \phi_k = 0$ ,  $k \geq 1$ .

**Example 3.** If  $\{X_k\}_{k=1}^\infty$  is a martingale difference sequence with finite forth moments, then (3) holds for  $\varphi_k = 0$ ,  $k \geq 1$ .

**Example 4.** Let  $\{X_k\}_{k=1}^\infty$  be centred, orthonormal, strongly mixing random variables with mixing coefficients  $(\alpha_k)_{k=1}^\infty$ . If these variables have uniformly bounded moments of order  $4\delta$  for some  $\delta > 1$ , then (3) and (4) hold with

$$\varphi_k = \phi_k = C\alpha_k^{(\delta-1)/\delta}$$

for large enough  $C > 0$  (see Corollary A.2 in [9]). One can give similar bounds for many other weakly dependent sequences.

**Example 5.** If  $\{X_k\}_{k=1}^\infty$  are independent identically distributed random variables with  $\mathbb{E}X_k = 0$  and  $\mathbb{E}X_k^2 = 1$ , then (3) and (4) hold for  $\varphi_k = \phi_k = 0$ ,  $k \geq 1$ , as well as Lindeberg's condition (7) hold.

We now assume that  $\{X_k\}_{k=1}^\infty$  is a sequence of orthonormal variables. Denote by  $\mathcal{X}$  the set of all random variables

$$Y = \sum_{k=1}^{\infty} c_k X_k \quad \text{a.s.}$$

for some  $\{c_k\}_{k=1}^\infty$  with  $\sum_{k=1}^{\infty} c_k^2 < \infty$ , where the first series converges in mean square. Let also  $\mathcal{X}_p$  be the set of all  $\mathbb{R}^p$ -valued random vectors  $\mathbf{y}_p$  which entries belong to  $\mathcal{X}$ .

**Corollary 4.8.** *Let  $\mathbf{y}_p \in \mathcal{X}_p$ ,  $\Sigma_p = \mathbb{E}\mathbf{y}_p\mathbf{y}_p^\top$  and  $A_p$  be a real symmetric positive*

semi-definite  $p \times p$  matrix for  $p \geq 1$ . If (B2) holds, then

$$\mathbb{E}|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)|^2 \leq C(\Phi_0 + \Phi_1 + \Phi_2)\|A_p\|^2 \text{tr}(\Sigma_p^2)$$

for a universal constant  $C > 0$ . Moreover, if (A2) holds, then (A1) holds.

**Corollary 4.9.** *Let  $\mathbf{y}_p \in \mathcal{X}_p$ ,  $\Sigma_p = \mathbb{E}\mathbf{y}_p \mathbf{y}_p^\top$  and  $A_p$  be a real symmetric positive semi-definite  $p \times p$  matrix for some  $p \geq 1$ . If (B4) holds, then*

$$\mathbb{E}|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)| \leq Cb\|A_p\|\sqrt{\text{tr}(\Sigma_p^2)} + CL(b)\|A_p\|\text{tr}(\Sigma_p)$$

for a universal constant  $C > 0$  and  $b > 1$ , where

$$L(b) = \sup_{k \geq 1} \mathbb{E}|X_k^2 - 1|I(|X_k^2 - 1| > b^2).$$

Moreover, if (A2) holds,  $\text{tr}(\Sigma_p) = O(p)$  as  $p \rightarrow \infty$  and  $\{X_k^2\}_{k=1}^\infty$  is a uniformly integrable family, then (A1) holds.

**Remark 6.** Papers [4], [15], [17] and [22] contain the universality principle for some  $\mathbf{y}_p \in \mathcal{X}_p$  when  $\{X_k\}_{k=1}^\infty$  are independent identically distributed random variables with  $\mathbb{E}X_k = 0$ ,  $\mathbb{E}X_k^2 = 1$ . This could be derived from the general universality principle given in the present paper.

## 5 Proofs

**Proof of Theorem 2.2.** Fix  $z \in \mathbb{C}^+$ . First we proceed as in Step 1 of the proof of Theorem 1.1 in [2] (see also the proof of (4.5.6) on page 83 in [1]) to show that

$S_n(z) - \mathbb{E}S_n(z) \rightarrow 0$  a.s. as  $n \rightarrow \infty$ , where

$$S_n(z) = p^{-1} \text{tr} \left( n^{-1} \mathbf{Y}_{pn} \mathbf{Y}_{pn}^\top - z I_p \right)^{-1}.$$

Similar arguments yield that  $s_n(z) - \mathbb{E}s_n(z) \rightarrow 0$  a.s. as  $n \rightarrow \infty$ , where

$$s_n(z) = p^{-1} \text{tr} \left( n^{-1} \mathbf{Z}_{pn} \mathbf{Z}_{pn}^\top - z I_p \right)^{-1}.$$

Hence, we only need to show that  $\mathbb{E}S_n(z) - \mathbb{E}s_n(z) \rightarrow 0$ .

We will use Lindeberg's method as in the proof of Theorem 6.1 in [7]. Let  $\mathbf{y}_{p1}, \dots, \mathbf{y}_{pn}$  and  $\mathbf{z}_{p1}, \dots, \mathbf{z}_{pn}$  be columns of  $\mathbf{Y}_{pn}$  and  $\mathbf{Z}_{pn}$  correspondingly. Assume also w.l.o.g. that  $\{(\mathbf{y}_{pk}, \Sigma_{pk})\}_{k=1}^n$  are independent copies of  $(\mathbf{y}_p, \Sigma_p)$  and  $\mathbf{z}_{pk} = \Sigma_{pk}^{1/2} \mathbf{w}_{pk}$  for all  $1 \leq k \leq n$ , where  $\{\mathbf{w}_{pk}\}_{k=1}^n$  are independent standard normal vectors in  $\mathbb{R}^p$  that are also independent of  $\{(\mathbf{y}_{pk}, \Sigma_{pk})\}_{k=1}^n$ .

Recall that

$$\mathbf{Y}_{pn} \mathbf{Y}_{pn}^\top = \sum_{k=1}^n \mathbf{y}_{pk} \mathbf{y}_{pk}^\top \quad \text{and} \quad \mathbf{Z}_{pn} \mathbf{Z}_{pn}^\top = \sum_{k=1}^n \mathbf{z}_{pk} \mathbf{z}_{pk}^\top.$$

Using this representation, we derive that

$$\begin{aligned} S_n(z) &= \frac{1}{p} \text{tr} \left( n^{-1} \sum_{k=1}^n \mathbf{y}_{pk} \mathbf{y}_{pk}^\top - z I_p \right)^{-1}, \\ s_n(z) &= \frac{1}{p} \text{tr} \left( n^{-1} \sum_{k=1}^n \mathbf{z}_{pk} \mathbf{z}_{pk}^\top - z I_p \right)^{-1} \end{aligned}$$

and  $|S_n(z) - s_n(z)| \leq \sum_{k=1}^n |I_{kn}|/p$ , where

$$I_{kn} = \text{tr} \left( C_{kn} + \frac{\mathbf{y}_{pk} \mathbf{y}_{pk}^\top}{n} - z I_p \right)^{-1} - \text{tr} \left( C_{kn} + \frac{\mathbf{z}_{pk} \mathbf{z}_{pk}^\top}{n} - z I_p \right)^{-1}$$



for  $C_{1n} = \sum_{i=2}^n \mathbf{z}_{pi} \mathbf{z}_{pi}^\top / n$ ,  $C_{nn} = \sum_{i=1}^{n-1} \mathbf{y}_{pi} \mathbf{y}_{pi}^\top / n$  and

$$C_{kn} = \frac{1}{n} \sum_{i=1}^{k-1} \mathbf{y}_{pi} \mathbf{y}_{pi}^\top + \frac{1}{n} \sum_{i=k+1}^n \mathbf{z}_{pi} \mathbf{z}_{pi}^\top, \quad 1 < k < n.$$

By the Sherman-Morrison-Woodbury formula,

$$\text{tr}(C + ww^\top - zI_p)^{-1} - \text{tr}(C - zI_p)^{-1} = -\frac{w^\top (C - zI_p)^{-2} w}{1 + w^\top (C - zI_p)^{-1} w}$$

for any real symmetric  $p \times p$  matrix  $C$  and  $w \in \mathbb{R}^p$ . Hence, adding and subtracting  $\text{tr}(C_{kn} - zI_p)^{-1}$  to  $I_{kn}$  yield

$$I_{kn} = -\frac{\mathbf{y}_{pk}^\top A_{kn}^2 \mathbf{y}_{pk} / n}{1 + \mathbf{y}_{pk}^\top A_{kn} \mathbf{y}_{pk} / n} + \frac{\mathbf{z}_{pk}^\top A_{kn}^2 \mathbf{z}_{pk} / n}{1 + \mathbf{z}_{pk}^\top A_{kn} \mathbf{z}_{pk} / n},$$

where we set  $A_{kn} = A_{kn}(z) = (C_{kn} - zI_p)^{-1}$ ,  $1 \leq k \leq n$ .

Let us now show that

$$\frac{1}{p} \sum_{k=1}^n \mathbb{E} |I_{kn}| \rightarrow 0.$$

The latter implies that  $\mathbb{E} S_n(z) - \mathbb{E} s_n(z) \rightarrow 0$ . To estimate  $I_{kn}$  we need the following lemma which proof is given in Appendix.

**Lemma 5.1.** *Let  $w \in \mathbb{R}^p$ ,  $C$  be a  $p \times p$  real symmetric matrix and  $z \in \mathbb{C}$  with  $\text{Im}(z) > 0$ . Then*

$$\frac{|w^\top (C - zI_p)^{-2} w|}{|1 + w^\top (C - zI_p)^{-1} w|} \leq \frac{1}{\text{Im}(z)}.$$

Write  $z = u + iv$  for  $u \in \mathbb{R}$  and  $v = \text{Im}(z) > 0$ . By Lemma 5.1,  $|I_{kn}| \leq 2/v$ .

Denote  $x \wedge y = \min\{x, y\}$ . Using inequalities

$$|x + y| \wedge 1 \leq (|x| + |y|) \wedge 1 \leq |x| \wedge 1 + |y| \wedge 1, \quad x, y \in \mathbb{C}, \quad (8)$$

we derive that

$$\mathbb{E}|I_{kn}| = \mathbb{E}(|I_{kn}| \wedge (2/v)) \leq \mathbb{E}(|\Delta_{kn}| \wedge (2/v)) + \mathbb{E}(|\hat{\Delta}_{kn}| \wedge (2/v))$$

where

$$\begin{aligned}\Delta_{kn} &= \frac{\mathbf{y}_{pk}^\top A_{kn}^2 \mathbf{y}_{pk}/n}{1 + \mathbf{y}_{pk}^\top A_{kn} \mathbf{y}_{pk}/n} - \frac{\text{tr}(\Sigma_{pk} A_{kn}^2)/n}{1 + \text{tr}(\Sigma_{pk} A_{kn})/n}, \\ \hat{\Delta}_{kn} &= \frac{\mathbf{z}_{pk}^\top A_{kn}^2 \mathbf{z}_{pk}/n}{1 + \mathbf{z}_{pk}^\top A_{kn} \mathbf{z}_{pk}/n} - \frac{\text{tr}(\Sigma_{pk} A_{kn}^2)/n}{1 + \text{tr}(\Sigma_{pk} A_{kn})/n}.\end{aligned}$$

We estimate only the term  $\mathbb{E}(|\Delta_{kn}| \wedge (2/v))$ , since  $\mathbb{E}(|\hat{\Delta}_{kn}| \wedge (2/v))$  can be estimated similarly (via Proposition 2.1).

Fix any  $k \in \{1, \dots, n\}$ . For notational simplicity, we will further write

$$\mathbf{y}_p, A_n, \Delta_n, C_n, \Sigma_p \quad \text{instead of} \quad \mathbf{y}_{pk}, A_{kn}, \Delta_{kn}, C_{kn}, \Sigma_{pk}$$

and use the following properties:  $C_n$  is a real symmetric positive semi-definite  $p \times p$  random matrix,  $(\mathbf{y}_p, \Sigma_p)$  is independent of  $A_n = (C_n - zI_p)^{-1}$ .

**Lemma 5.2.** *Let  $A = (C - zI_p)^{-1}$ , where  $C$  is a real symmetric positive semi-definite  $p \times p$  matrix and  $z \in \mathbb{C}^+$ . Then the spectral norm of  $A$  satisfies  $\|A\| \leq 1/\text{Im}(z)$ .*

**Lemma 5.3.** *Let (A1) holds. Then, for each  $\varepsilon, M > 0$ ,*

$$\lim_{p \rightarrow \infty} \sup_{A_p} \mathbb{P}(|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)| > \varepsilon p) = 0, \quad (9)$$

where the supremum is taken over all complex  $p \times p$  matrices  $A_p$  with  $\|A_p\| \leq M$ .

Fix any  $\varepsilon > 0$ . Take

$$D_n = \bigcap_{j=1}^2 \{|\mathbf{y}_p^\top (A_n)^j \mathbf{y}_p - \text{tr}(\Sigma_p(A_n)^j)| \leq \varepsilon p\}$$

and derive that

$$\mathbb{E}(|\Delta_n| \wedge (2/v)) \leq \mathbb{E}(|\Delta_n| \wedge (2/v))I(D_n) + 2\mathbb{P}(\overline{D}_n)/v.$$

By the law of iterated mathematical expectations and Lemma 5.2,

$$\mathbb{P}(\overline{D}_n) = \mathbb{E}[\mathbb{P}(\overline{D}_n | A_n)] \leq 2 \sup_{\hat{A}_p} \mathbb{P}(|\mathbf{y}_p^\top \hat{A}_p \mathbf{y}_p - \text{tr}(\Sigma_p \hat{A}_p)| > \varepsilon p),$$

where  $\hat{A}_p$  are as in Lemma 5.3 with  $M = \max\{v^{-1}, v^{-2}\}$ .

**Lemma 5.4.** *Let  $z_1, z_2, w_1, w_2 \in \mathbb{C}$ . If  $|z_1 - z_2| \leq \gamma$ ,  $|w_1 - w_2| \leq \gamma$ ,*

$$\frac{|z_1|}{|1 + w_1|} \leq M$$

*and  $|1 + w_2| \geq \delta$  for some  $\delta, M > 0$  and  $\gamma \in (0, \delta/2)$ , then*

$$\left| \frac{z_1}{1 + w_1} - \frac{z_2}{1 + w_2} \right| \leq C\gamma$$

*for some  $C = C(\delta, M) > 0$ .*

**Lemma 5.5.** *Let  $z \in \mathbb{C}^+$ ,  $\Sigma$  and  $C$  be real symmetric positive semi-definite  $p \times p$  matrices. Then*

$$|1 + \text{tr}(\Sigma(C - zI_p)^{-1})| \geq \frac{\text{Im}(z)}{|z|}.$$

By Lemma 5.5, we get

$$|1 + \text{tr}(\Sigma_p A_n)/n| = |1 + \text{tr}((\Sigma_p/n)(C_n - zI_p)^{-1})| \geq \delta = \frac{v}{|z|}. \quad (10)$$

Take  $\gamma = \varepsilon p/n$ ,

$$(z_1, w_1) = (\mathbf{y}_p^\top A_n^2 \mathbf{y}_p, \mathbf{y}_p^\top A_n \mathbf{y}_p)/n, \quad (z_2, w_2) = (\text{tr}(\Sigma_p A_n^2), \text{tr}(\Sigma_p A_n))/n$$

in Lemma 5.4. By Lemma 5.1,

$$\frac{|z_1|}{|1 + w_1|} \leq \frac{1}{v}.$$

By (10),

$$|1 + w_2| \geq \delta > 3\varepsilon y > \frac{2\varepsilon p}{n}$$

for small enough  $\varepsilon > 0$  and large enough  $p$  (since  $p/n \rightarrow y > 0$ ).

Using Lemma 5.4, we derive

$$\mathbb{E}(|\Delta_n| \wedge (2/v)) I(D_n) \leq \mathbb{E}|\Delta_n| I(D_n) \leq C(\delta, 1/v) \frac{\varepsilon p}{n}.$$

Combining all above estimates together yields

$$\mathbb{E}(|\Delta_{kn}| \wedge (2/v)) \leq C(\delta, 1/v) \frac{\varepsilon p}{n} + \frac{4}{v} \sup_{\hat{A}_p} \mathbb{P}(|\mathbf{y}_p^\top \hat{A}_p \mathbf{y}_p - \text{tr}(\Sigma_p \hat{A}_p)| > \varepsilon p)$$

for each  $k = 1, \dots, n$  and

$$\frac{1}{p} \sum_{k=1}^n \mathbb{E}(|\Delta_{kn}| \wedge (2/v)) \leq C(\delta, 1/v) \varepsilon + \frac{4n}{vp} \sup_{\hat{A}_p} \mathbb{P}(|\mathbf{y}_p^\top \hat{A}_p \mathbf{y}_p - \text{tr}(\Sigma_p \hat{A}_p)| > \varepsilon p).$$

Taking  $\varepsilon$  small enough and then  $p$  large enough, we can make the right hand side of

the last inequality arbitrarily small by Lemma 5.3 (recall also that  $n/p \rightarrow 1/y > 0$ ).

Thus,

$$\lim_{n \rightarrow \infty} \frac{1}{p} \sum_{k=1}^n \mathbb{E}(|\Delta_{kn}| \wedge (2/v)) = 0.$$

Arguing as above with the help of Proposition 2.1, one can prove that

$$\lim_{n \rightarrow \infty} \frac{1}{p} \sum_{k=1}^n \mathbb{E}(|\hat{\Delta}_{kn}| \wedge (2/v)) = 0.$$

This finishes the proof. Q.e.d.

**Proof of Proposition 2.3.** Using the Vitali convergence theorem (see Lemma 2.14 on page 37 in [1]), one can prove that

$$\mathbb{P}(s_n(z) \rightarrow s(z) \quad \text{for all } z \in \mathbb{C}^+) = 1, \quad (11)$$

where  $s_n(z) = p^{-1} \text{tr}(A_n - zI_p)^{-1}$ . For a proof, see Step 3 of the proof of Theorem 2.9 on page 37 in [1]. Having (11), it is easy to finish the proof by applying Theorem B.9 in [1]. Q.e.d.

**Proof of Theorem 2.4.** The result follows from Theorem 2.2 and Proposition 2.3 as well as Theorem 1.1 in [2]. Q.e.d.

**Proof of Theorem 3.1.** The proof is along the proof of Theorem 2.2. However, additional arguments are needed. In what follows, we will use the same notations as in the proof of Theorem 2.2.

First, to prove that  $S_n(z) - \mathbb{E}S_n(z) \rightarrow 0$  a.s., we use results of [5]. Namely, by A.2 in Chapter 9 of [14], non-zero eigenvalues of

$$\mathbf{X}_{np} = \frac{1}{\sqrt{p+n}} \begin{pmatrix} O_p & \mathbf{Y}_{pn} \\ \mathbf{Y}_{pn}^\top & O_n \end{pmatrix}$$

are square roots of non-zero eigenvalues of  $n^{-1}\mathbf{Y}_{pn}\mathbf{Y}_{pn}^\top$  multiplied by  $n/(p+n)$  and their negatives, where  $O_k$  is the  $k \times k$  zero matrix for  $k \geq 1$ . If the second matrix has exactly  $j$  zero eigenvalues, then

$$S_n(z) = \frac{1}{p} \sum_{k=1}^{p-j} \frac{1}{\lambda_{kn} - z} - \frac{j}{zp} = \frac{1}{2p\sqrt{z}} \sum_{k=1}^{p-j} \left[ \frac{1}{\sqrt{\lambda_{kn}} - \sqrt{z}} + \frac{1}{-\sqrt{\lambda_{kn}} - \sqrt{z}} \right] - \frac{j}{zp},$$

where  $\lambda_{kn}$ ,  $k = 1, \dots, p-j$ , are non-zero eigenvalues of  $n^{-1}\mathbf{Y}_{pn}\mathbf{Y}_{pn}^\top$  and  $\sqrt{z}$  is chosen in a way that  $\sqrt{z} \in \mathbb{C}^+$  for  $z \in \mathbb{C}^+$ . Define

$$\widehat{S}_n(w) = \frac{1}{p+n} \text{tr}(\mathbf{X}_{np} - wI_{p+n})^{-1}, \quad w \in \mathbb{C}^+.$$

Then

$$\widehat{S}_n(w) = \frac{1}{p+n} \sum_{k=1}^{p-j} \left[ \frac{1}{\sqrt{n\lambda_{kn}/(p+n)} - w} + \frac{1}{-\sqrt{n\lambda_{kn}/(p+n)} - w} \right] - \frac{n-p+2j}{w(p+n)}.$$

as well as

$$S_n(z) = \frac{\sqrt{n(p+n)}}{2p\sqrt{z}} \widehat{S}_n(w_n) + \frac{n-p}{2zp}, \quad w_n = \sqrt{nz/(p+n)}.$$

We now apply Proposition 12 from [5] (see also (41) in [5])

$$\begin{aligned} \mathbb{P}(|S_n(z) - \mathbb{E}S_n(z)| > \varepsilon) &= \mathbb{P}(|\widehat{S}_n(w_n) - \mathbb{E}\widehat{S}_n(w_n)| > 2\varepsilon p\sqrt{|z|}/\sqrt{n(p+n)}) \\ &\leq \exp \left\{ -\frac{(p+n)|\text{Im}(w_n)|^2}{2560(m+1)} \cdot \frac{4\varepsilon^2 p^2 |z|}{n(p+n)} \right\} \\ &= \exp \left\{ -\frac{n|\text{Im}(\sqrt{z})|^2}{2560(m+1)} \cdot \frac{4\varepsilon^2 p^2 |z|}{n(p+n)} \right\} \\ &= \exp \left\{ -\frac{Kn}{m+1} \right\} \end{aligned}$$

with

$$K = K(n) = \frac{4\varepsilon^2 y^2 |z \operatorname{Im}(\sqrt{z})|^2}{2560(y+1)} + o(1), \quad n \rightarrow \infty.$$

If  $m = m(n) = o(n/\log n)$  as  $n \rightarrow \infty$ , the latter and the Borel-Cantelli lemma imply that  $S_n(z) - \mathbb{E}S_n(z) \rightarrow 0$  a.s. Analogously, we obtain that  $s_n(z) - \mathbb{E}s_n(z) \rightarrow 0$  a.s.

Then we proceed using the same arguments as in the proof of Theorem 2.2 till the definition of  $I_{kn}$ , i.e.

$$I_{kn} = -\frac{\mathbf{y}_{pk}^\top A_{kn}^2 \mathbf{y}_{pk}/n}{1 + \mathbf{y}_{pk}^\top A_{kn} \mathbf{y}_{pk}/n} + \frac{\mathbf{z}_{pk}^\top A_{kn}^2 \mathbf{z}_{pk}/n}{1 + \mathbf{z}_{pk}^\top A_{kn} \mathbf{z}_{pk}/n},$$

where  $A_{kn} = A_{kn}(z) = (C_{kn} - zI_p)^{-1}$ ,  $1 \leq k \leq n$ . As in the proof of Theorem 2.2, we finish the proof if we show that

$$\frac{1}{p} \sum_{k=1}^n \mathbb{E}|I_{kn}| \rightarrow 0. \quad (12)$$

Applying (8), we arrive at

$$\mathbb{E}|I_{kn}| = \mathbb{E}(|I_{kn}| \wedge (2/v)) \leq \mathbb{E}(|\Delta_{kn}^m| \wedge (2/v)) + \mathbb{E}(|\widehat{\Delta}_{kn}^m| \wedge (2/v)),$$

where

$$\begin{aligned} \Delta_{kn}^m &= \frac{\mathbf{y}_{pk}^\top A_{kn}^2 \mathbf{y}_{pk}/n}{1 + \mathbf{y}_{pk}^\top A_{kn} \mathbf{y}_{pk}/n} - \frac{\operatorname{tr}(\Sigma_{pk}(A_{kn}^m)^2)/n}{1 + \operatorname{tr}(\Sigma_{pk} A_{kn}^m)/n}, \\ \widehat{\Delta}_{kn}^m &= \frac{\mathbf{z}_{pk}^\top A_{kn}^2 \mathbf{z}_{pk}/n}{1 + \mathbf{z}_{pk}^\top A_{kn} \mathbf{z}_{pk}/n} - \frac{\operatorname{tr}(\Sigma_{pk}(A_{kn}^m)^2)/n}{1 + \operatorname{tr}(\Sigma_{pk} A_{kn}^m)/n}. \end{aligned}$$

Here  $A_{kn}^m = A_{kn}^m(z) = (C_{kn}^m - zI_p)^{-1}$  and

$$C_{kn}^m = C_{kn} - \frac{1}{n} \sum_{i=k-2m+1}^{k-1} \mathbf{y}_{pi} \mathbf{y}_{pi}^\top - \frac{1}{n} \sum_{i=k+1}^{k+2m-1} \mathbf{z}_{pi} \mathbf{z}_{pi}^\top, \quad 1 \leq k \leq n,$$

where, for simplicity, we let  $\mathbf{y}_{pi}$  and  $\mathbf{z}_{pi}$  be zero vectors for all  $i$  such that

$$-2m + 2 \leq i \leq 0 \quad \text{or} \quad n + 1 \leq i \leq n + 2m - 1.$$

Note that  $C_{kn}^m$  is independent of  $(\mathbf{y}_{pk}, \mathbf{z}_{pk})$  for each  $k = 1, \dots, n$  because of the  $m$ -dependence in  $\{(\mathbf{y}_{pk}, \mathbf{z}_{pk})\}_{k=-2m+2}^{n+2m-1}$ .

It is shown in the proof of Theorem 2.2 that, for any  $\varepsilon > 0$  and  $z \in \mathbb{C}^+$ ,

$$\frac{1}{p} \sum_{k=1}^n \mathbb{E}(|\Delta_{kn}^m| \wedge (2/v)) \leq C(v/|z|, 1/v)\varepsilon + \frac{2}{vp} \sum_{k=1}^n \mathbb{P}(E_{kn}^m),$$

where  $C = C(a, b) > 0$  is given in Lemma 5.4,  $v = \text{Im}(z) > 0$  and events  $E_{kn}^m = E_{kn}^m(\varepsilon)$  are defined by

$$E_{kn}^m = \bigcup_{j=1}^2 \{|\mathbf{y}_{pk}^\top (A_{kn})^j \mathbf{y}_{pk} - \text{tr}(\Sigma_{pk}(A_{kn}^m)^j)| > \varepsilon p\}.$$

A similar bound is valid when  $\mathbb{E}(|\Delta_{kn}^m| \wedge (2/v))$  is replaced by  $\mathbb{E}(|\widehat{\Delta}_{kn}^m| \wedge (2/v))$ , i.e.

$$\frac{1}{p} \sum_{k=1}^n \mathbb{E}(|\widehat{\Delta}_{kn}^m| \wedge (2/v)) \leq C(v/|z|, 1/v)\varepsilon + \frac{2}{vp} \sum_{k=1}^n \mathbb{P}(\widehat{E}_{kn}^m),$$

where

$$\widehat{E}_{kn}^m = \widehat{E}_{kn}^m(\varepsilon) = \bigcup_{j=1}^2 \{|\mathbf{z}_{pk}^\top (A_{kn})^j \mathbf{z}_{pk} - \text{tr}(\Sigma_{pk}(A_{kn}^m)^j)| > \varepsilon p\}.$$

Thus, to verify (12), we need to prove that (for any fixed  $\varepsilon > 0$ )

$$\frac{1}{p} \sum_{k=1}^n [\mathbb{P}(E_{kn}^m) + \mathbb{P}(\widehat{E}_{kn}^m)] \rightarrow 0.$$



Fix any  $\varepsilon > 0$ . Obviously,  $E_{kn}^m \subseteq F_{kn}^m \cup G_{kn}^m$  for

$$F_{kn}^m = \bigcup_{j=1}^2 \{|\mathbf{y}_{pk}^\top (A_{kn})^j \mathbf{y}_{pk} - \mathbf{y}_{pk}^\top (A_{kn}^m)^j \mathbf{y}_{pk}| > \varepsilon p/2\},$$

$$G_{kn}^m = \bigcup_{j=1}^2 \{|\mathbf{y}_{pk}^\top (A_{kn}^m)^j \mathbf{y}_{pk} - \text{tr}(\Sigma_{pk}(A_{kn}^m)^j)| > \varepsilon p/2\}.$$

**Lemma 5.6.** *If (A3) holds, then, for any  $\varepsilon, M > 0$ ,*

$$\frac{1}{n} \sum_{k=1}^n \sup_{A_p} \mathbb{P}(|\mathbf{y}_{pk}^\top A_p \mathbf{y}_{pk} - \text{tr}(\Sigma_{pk} A_p)| > \varepsilon p) \rightarrow 0, \quad n \rightarrow \infty,$$

where all suprema are taken over all complex  $p \times p$  matrices  $A_p$  with  $\|A_p\| \leq M$ .

Using independence of  $\mathbf{y}_{pk}$  and  $(A_{kn}^m)^j$ , we infer from Lemmas 5.2 and 5.6 that

$$\frac{1}{p} \sum_{k=1}^n \mathbb{P}(G_{kn}^m) \leq \frac{2}{p} \sum_{k=1}^n \sup_{A_p} \mathbb{P}(|\mathbf{y}_{pk}^\top A_p \mathbf{y}_{pk} - \text{tr}(\Sigma_{pk} A_p)| > \varepsilon p/2) \rightarrow 0, \quad n \rightarrow \infty,$$

where each suprema is taken over all complex  $p \times p$  matrices  $A_p$  with spectral norms

$\|A_p\| \leq M = 1/(v \wedge v^2)$  and we have used the fact that  $p/n \rightarrow y > 0$ .

Similarly,  $\widehat{E}_{kn}^m \subseteq \widehat{F}_{kn}^m \cup \widehat{G}_{kn}^m$  for

$$\widehat{F}_{kn}^m = \bigcup_{j=1}^2 \{|\mathbf{z}_{pk}^\top (A_{kn})^j \mathbf{z}_{pk} - \mathbf{z}_{pk}^\top (A_{kn}^m)^j \mathbf{z}_{pk}| > \varepsilon p/2\},$$

$$\widehat{G}_{kn}^m = \bigcup_{j=1}^2 \{|\mathbf{z}_{pk}^\top (A_{kn}^m)^j \mathbf{z}_{pk} - \text{tr}(\Sigma_{pk}(A_{kn}^m)^j)| > \varepsilon p/2\}.$$

By independence of  $\mathbf{z}_{pk}$  and  $(A_{kn}^m)^j$ , we derive from Lemma 5.2 that

$$\frac{1}{p} \sum_{k=1}^n \mathbb{P}(\widehat{G}_{kn}^m) \leq \frac{2}{p} \sum_{k=1}^n \sup_{A_p} \mathbb{P}(|\mathbf{z}_{pk}^\top A_p \mathbf{z}_{pk} - \text{tr}(\Sigma_{pk} A_p)| > \varepsilon p/2),$$

where each suprema is taken over all complex  $p \times p$  matrices  $A_p$  with spectral norms  $\|A_p\| \leq M = 1/(v \wedge v^2)$ . It is shown in the proof of Proposition 2.1 that

$$\mathbb{P}(|\mathbf{z}_{pk}^\top A_p \mathbf{z}_{pk} - \text{tr}(\Sigma_{pk} A_p)| > \varepsilon p/2) \leq \mathbb{E} \min \left\{ \frac{16M^2 \text{tr}(\Sigma_{pk}^2)}{(\varepsilon p/2)^2}, 1 \right\}.$$

To show that

$$\frac{1}{p} \sum_{k=1}^n \mathbb{P}(\widehat{G}_{kn}^m) \rightarrow 0,$$

we only need to note that (A4) implies that, for any  $\varepsilon, M > 0$ ,

$$\frac{1}{p} \sum_{k=1}^n \mathbb{E} \min \left\{ \frac{16M^2 \text{tr}(\Sigma_{pk}^2)}{(\varepsilon p/2)^2}, 1 \right\} \rightarrow 0. \quad (13)$$

Indeed, by (A4), for any  $\gamma > 0$ ,

$$\begin{aligned} \frac{1}{p} \sum_{k=1}^n \mathbb{E} \min \left\{ \frac{16M^2 \text{tr}(\Sigma_{pk}^2)}{(\varepsilon p/2)^2}, 1 \right\} &\leq \frac{n}{p} \frac{64M^2 \gamma}{\varepsilon^2} + \frac{1}{p} \sum_{k=1}^n \mathbb{P}(\text{tr}(\Sigma_{pk}^2) > \gamma p^2) \\ &= \frac{64M^2 \gamma}{y \varepsilon^2} + o(1). \end{aligned}$$

This clearly implies (13). Additionally, by the Markov inequality,

$$\frac{1}{p} \sum_{k=1}^n [\mathbb{P}(F_{kn}^m) + \mathbb{P}(\widehat{F}_{kn}^m)] \leq \frac{2}{\varepsilon p^2} \sum_{k=1}^n J_k$$

with

$$J_k = \sum_{j=1}^2 (\mathbb{E} |\mathbf{y}_{pk}^\top (A_{kn})^j \mathbf{y}_{pk} - \mathbf{y}_{pk}^\top (A_{kn}^m)^j \mathbf{y}_{pk}| + \mathbb{E} |\mathbf{z}_{pk}^\top (A_{kn})^j \mathbf{z}_{pk} - \mathbf{z}_{pk}^\top (A_{kn}^m)^j \mathbf{z}_{pk}|).$$

To finish the proof of the theorem, we need to show that

$$\frac{1}{p^2} \sum_{k=1}^n J_k \rightarrow 0.$$

We need three additional lemmas.

**Lemma 5.7.** *Let  $z \in \mathbb{C}^+$ ,  $U$  be a real  $p \times q$  matrix and  $C$  be a real symmetric positive semi-definite  $p \times p$  matrix. Then*

$$\|(I_q + U^\top (C - zI_p)^{-1} U)^{-1}\| \leq \frac{|z|}{\text{Im}(z)}.$$

**Lemma 5.8.** *Let  $z \in \mathbb{C}^+$ ,  $w \in \mathbb{C}^q$ ,  $U$  be a real  $p \times q$  matrix and  $C$  be a real symmetric positive semi-definite  $p \times p$  matrix. Then*

$$|w^\top A w| \leq \frac{\text{Im}(z) + |z|}{|\text{Im}(z)|^2} \|w\|^2,$$

where  $A = (I_q + U^\top (C - zI_p)^{-1} U)^{-1} U^\top (C - zI_p)^{-2} U (I_q + U^\top (C - zI_p)^{-1} U)^{-1}$ .

**Lemma 5.9.** *Let  $z \in \mathbb{C}^+$ ,  $y \in \mathbb{C}^p$ ,  $U$  be a real  $p \times q$  matrix and  $C$  be a real symmetric positive semi-definite  $p \times p$  matrix. Then*

$$\sum_{j=1}^2 |y^\top (C + U U^\top - zI_p)^{-j} y - y^\top (C - zI_p)^{-j} y| \leq \frac{2(|z| + 1)^2}{|\text{Im}(z)|^2} \sum_{j=1}^2 \|U^\top (C - zI_p)^{-j} y\|^2.$$

Taking  $C = C_{kn}^m$  and  $U$  to be  $p \times (4m - 2)$  matrix with columns

$$n^{-1/2} \mathbf{y}_{pi}, \quad k - 2m + 1 \leq i \leq k - 1, \quad \text{and} \quad n^{-1/2} \mathbf{z}_{pi}, \quad k + 1 \leq i \leq k + 2m - 1,$$

in Lemma 5.9, we get  $A_{kn} = (C + UU^\top - zI_p)^{-1}$  and  $A_{kn}^m = (C - zI_p)^{-1}$  as well as

$$\begin{aligned} \sum_{j=1}^2 \mathbb{E} |\mathbf{y}_{pk}^\top (A_{kn})^j \mathbf{y}_{pk} - \mathbf{y}_{pk}^\top (A_{kn}^m)^j \mathbf{y}_{pk}| &\leq \frac{2(|z|+1)^2}{v^2} \sum_{j=1}^2 \mathbb{E} \|U^\top (C - zI_p)^{-j} \mathbf{y}_{pk}\|^2 \\ &= \frac{2(|z|+1)^2}{v^2 n} \sum_{j=1}^2 Q_{kj}, \end{aligned}$$

where

$$Q_{kj} = \sum_{i=k-2m+1}^{k-1} \mathbb{E} |\mathbf{y}_{pi}^\top (A_{kn}^m)^j \mathbf{y}_{pk}|^2 + \sum_{i=k+1}^{k+2m-1} \mathbb{E} |\mathbf{z}_{pi}^\top (A_{kn}^m)^j \mathbf{y}_{pk}|^2, \quad j = 1, 2.$$

By the same arguments,

$$\sum_{j=1}^2 \mathbb{E} |\mathbf{z}_{pk}^\top (A_{kn})^j \mathbf{z}_{pk} - \mathbf{z}_{pk}^\top (A_{kn}^m)^j \mathbf{z}_{pk}| \leq \frac{2(|z|+1)^2}{v^2 n} \sum_{j=1}^2 R_{kj},$$

where

$$R_{kj} = \sum_{i=k-2m+1}^{k-1} \mathbb{E} |\mathbf{y}_{pi}^\top (A_{kn}^m)^j \mathbf{z}_{pk}|^2 + \sum_{i=k+1}^{k+2m-1} \mathbb{E} |\mathbf{z}_{pi}^\top (A_{kn}^m)^j \mathbf{z}_{pk}|^2, \quad j = 1, 2.$$

Let us estimate the sums  $Q_{kj}$  and  $R_{kj}$ . There are two types of terms in  $Q_{kj}$  and  $R_{kj}$ . Namely, terms

$$\mathbb{E} |\mathbf{y}_{pi}^\top (A_{kn}^m)^j \mathbf{z}_{pk}|^2 \quad \text{and} \quad \mathbb{E} |\mathbf{y}_{pi}^\top (A_{kn}^m)^j \mathbf{y}_{pk}|^2, \quad k - 2m + 1 \leq i \leq k - m,$$

as well as

$$\mathbb{E} |\mathbf{z}_{pi}^\top (A_{kn}^m)^j \mathbf{z}_{pk}|^2 \quad \text{and} \quad \mathbb{E} |\mathbf{z}_{pi}^\top (A_{kn}^m)^j \mathbf{y}_{pk}|^2, \quad k + m \leq i \leq k + 2m - 1,$$

have the form  $\mathbb{E}|x_1^\top Ax_2|^2$  for a  $p \times p$  symmetric random matrix  $A$  and random vectors  $x_1, x_2$  in  $\mathbb{R}^p$  such that  $(A, x_1)$  is independent of  $x_2$ . The rest terms have the form  $\mathbb{E}|x_1^\top Ax_2|^2$  with  $(x_1, x_2)$  independent of  $A$ . In all cases, the spectral norm of  $A$  is almost surely bounded by  $M = \max\{v^{-1}, v^{-2}\}$  (see Lemma 5.2).

Let  $A$  be any complex symmetric random  $p \times p$  matrix that is independent of  $(\mathbf{y}_{pj}, \mathbf{y}_{pl}, \mathbf{z}_{pj}, \mathbf{z}_{pl})$ ,  $1 \leq j < l \leq n$  and  $\|A\| \leq M$  a.s. Set  $A^* = \overline{A}^\top = \overline{A}$  and  $\mathbb{E}_s = \mathbb{E}(\cdot | \mathcal{F}_s^p)$  for all  $0 \leq s \leq n-1$  for  $\mathcal{F}_s^p$  defined in the beginning of Section 3. By the construction of  $\mathbf{z}_{pl} = \Sigma_{pl}^{1/2} \mathbf{w}_{pl}$  and the law of iterated mathematical expectations,

$$\begin{aligned} \mathbb{E}|\mathbf{y}_{pj}^\top A \mathbf{z}_{pl}|^2 &= \mathbb{E} \mathbf{y}_{pj}^\top A^* \mathbf{z}_{pl} \mathbf{z}_{pl}^\top A \mathbf{y}_{pj} = \mathbb{E} \mathbf{y}_{pj}^\top A^* (\mathbb{E}_j \mathbf{z}_{pl} \mathbf{z}_{pl}^\top) A \mathbf{y}_{pj} = \\ &= \mathbb{E} \mathbf{y}_{pj}^\top A^* (\mathbb{E}_j \Sigma_{pl}) A \mathbf{y}_{pj} \leq \mathbb{E} \|\mathbb{E}_j \Sigma_{pl}\| \|A \mathbf{y}_{pj}\|^2 \leq M^2 \mathbb{E} \|\mathbb{E}_j \Sigma_{pl}\| \|\mathbf{y}_{pj}\|^2. \end{aligned}$$

Analogously,  $\mathbb{E}|\mathbf{y}_{pj}^\top A \mathbf{y}_{pl}|^2 \leq M^2 \mathbb{E} \|\mathbb{E}_j \mathbf{y}_{pl} \mathbf{y}_{pl}^\top\| \|\mathbf{y}_{pj}\|^2$  and

$$\begin{aligned} \mathbb{E}|\mathbf{z}_{pj}^\top A \mathbf{z}_{pl}|^2 &\leq M^2 \mathbb{E} \|\mathbb{E}_j \Sigma_{pl}\| \|\mathbf{z}_{pj}\|^2 = \\ &= M^2 \mathbb{E} \|\mathbb{E}_j \Sigma_{pl}\| \text{tr}(\Sigma_{pj}^{1/2} \mathbf{w}_{pj} \mathbf{w}_{pj}^\top \Sigma_{pj}^{1/2}) = M^2 \mathbb{E} \|\mathbb{E}_j \Sigma_{pl}\| \text{tr}(\Sigma_{pj}). \end{aligned}$$

Let  $(A, \mathbf{y}_{pj})$  be independent of  $(\mathbf{y}_{pl}, \mathbf{z}_{pl})$ , but  $A$  and  $\mathbf{y}_{pj}$  may be dependent. Then

$$\begin{aligned} \mathbb{E}|\mathbf{y}_{pj}^\top A \mathbf{z}_{pl}|^2 &= \mathbb{E} \mathbf{y}_{pj}^\top A^* \mathbf{z}_{pl} \mathbf{z}_{pl}^\top A \mathbf{y}_{pj} = \mathbb{E} \mathbf{y}_{pj}^\top A^* [\mathbb{E}(\Sigma_{pl}^{1/2} \mathbf{w}_{pl} \mathbf{w}_{pl}^\top \Sigma_{pl}^{1/2})] A \mathbf{y}_{pj} \leq \\ &\leq M^2 \|\mathbb{E} \Sigma_{pl}\| \mathbb{E} \|\mathbf{y}_{pj}\|^2 = M^2 \|\mathbb{E} \Sigma_{pl}\| \text{tr}(\mathbb{E} \mathbf{y}_{pj} \mathbf{y}_{pj}^\top). \end{aligned} \tag{14}$$

Similarly,  $\mathbb{E}|\mathbf{y}_{pj}^\top A \mathbf{y}_{pl}|^2 \leq M^2 \|\mathbb{E} \mathbf{y}_{pl} \mathbf{y}_{pl}^\top\| \mathbb{E} \|\mathbf{y}_{pj}\|^2$ .

Let  $(A, \mathbf{z}_{pl})$  be independent of  $(\mathbf{y}_{pj}, \mathbf{z}_{pj})$ , but  $A$  and  $\mathbf{z}_{pl}$  may be dependent. Then

$$\begin{aligned} \mathbb{E}|\mathbf{y}_{pj}^\top A \mathbf{z}_{pl}|^2 &= \mathbb{E}|\mathbf{z}_{pl}^\top A \mathbf{y}_{pj}|^2 = \mathbb{E}\mathbf{z}_{pl}^\top A^* \mathbf{y}_{pj} \mathbf{y}_{pj}^\top A \mathbf{z}_{pl} = \mathbb{E}\mathbf{z}_{pl}^\top A^* \mathbb{E}\mathbf{y}_{pj} \mathbf{y}_{pj}^\top A \mathbf{z}_{pl} \leq \\ &\leq M^2 \|\mathbb{E}\mathbf{y}_{pj} \mathbf{y}_{pj}^\top\| \|\mathbb{E}\mathbf{z}_{pj}\|^2 = M^2 \|\mathbb{E}\mathbf{y}_{pj} \mathbf{y}_{pj}^\top\| \text{tr}(\mathbb{E}\Sigma_{pj}). \end{aligned} \quad (15)$$

Similarly,  $\mathbb{E}|\mathbf{z}_{pj}^\top A \mathbf{z}_{pl}|^2 = \mathbb{E}|\mathbf{z}_{pl}^\top A \mathbf{z}_{pj}|^2 \leq M^2 \|\mathbb{E}\Sigma_{pj}\| \text{tr}(\mathbb{E}\Sigma_{pl})$ .

Combining obtained estimates, we have the following estimate

$$Q_{k1} + Q_{k2} + R_{k1} + R_{k2} \leq 2M^2(S_{k1} + S_{k2} + S_{k3} + S_{k4}),$$

where

$$\begin{aligned} S_{k1} &= \sum_{i: m \leq k-i < 2m} (\|\mathbb{E}\Sigma_{pk}\| + \|\mathbb{E}\mathbf{y}_{pk} \mathbf{y}_{pk}^\top\|) \mathbb{E}\|\mathbf{y}_{pi}\|^2 \\ S_{k2} &= \sum_{i: 1 \leq k-i < m} \mathbb{E}(\|\mathbb{E}_i \Sigma_{pk}\| + \|\mathbb{E}_i \mathbf{y}_{pk} \mathbf{y}_{pk}^\top\|) \|\mathbf{y}_{pi}\|^2 \\ S_{k3} &= \sum_{i: 1 \leq i-k < m} \mathbb{E}\|\mathbb{E}_k \Sigma_{pi}\| [\|\mathbf{y}_{pk}\|^2 + \text{tr}(\Sigma_{pk})] \\ S_{k4} &= \sum_{i: m \leq k-i < 2m} (\|\mathbb{E}\Sigma_{pk}\| + \|\mathbb{E}\mathbf{y}_{pk} \mathbf{y}_{pk}^\top\|) \text{tr}(\mathbb{E}\Sigma_{pi}). \end{aligned}$$

Define

$$T_{n1} = \sum \mathbb{E}(\|\mathbb{E}_k \Sigma_{pl}\| + \|\mathbb{E}_k \mathbf{y}_{pl} \mathbf{y}_{pl}^\top\|) \text{tr}(\mathbf{y}_{pk} \mathbf{y}_{pk}^\top + \Sigma_{pk})$$

with the sum taken over all  $(k, l)$  with  $1 \leq k < l \leq n$  and  $l < k + m$  and

$$T_{n2} = \sum (\|\mathbb{E}\Sigma_{pl}\| + \|\mathbb{E}\mathbf{y}_{pl} \mathbf{y}_{pl}^\top\|) \text{tr}(\mathbb{E}\mathbf{y}_{pk} \mathbf{y}_{pk}^\top + \mathbb{E}\Sigma_{pk})$$

with the sum taken over all  $(k, l)$  with  $1 \leq k, l \leq n$  and  $m \leq |k - l| < 2m$ .

The above estimates imply that

$$\frac{1}{p^2} \sum_{k=1}^n J_k \leq \frac{4(|z|+1)^2 M^2}{v^2 p^2 n} \sum_{k=1}^n \sum_{j=1}^2 (Q_{kj} + R_{kj}) \leq \frac{8(|z|+1)^2 M^2}{v^2 p^2 n} (T_{n1} + T_{n2}).$$

Using (A5) and  $n/p \rightarrow 1/y > 0$ , we get that

$$\frac{1}{p^2} \sum_{k=1}^n J_k \rightarrow 0.$$

This finishes the proof of the theorem. Q.e.d.

**Proof of Proposition 3.2.** Since there are  $O(nm)$  terms in each sum appearing in (A5), we will verify (A5) if we show that

$$\|\mathbb{E}_l \Sigma_{pk}\| \leq \mathbb{E}_l \|\mathbb{E}_{k-1} \Sigma_{pk}\| \quad \text{and} \quad \|\mathbb{E}_l \mathbf{y}_{pk} \mathbf{y}_{pk}^\top\| \leq \mathbb{E}_l \|\mathbb{E}_{k-1} \mathbf{y}_{pk} \mathbf{y}_{pk}^\top\|$$

for any  $0 \leq l < k \leq n$ . All these inequalities follow from Jensen's inequality and the convexity of the spectral norm (alternatively, see E.1.Theorem in [14]). Q.e.d.

**Proof of Theorem 3.3.** By the definition of  $L_n$ ,  $\widehat{\mathbf{Y}}_{pn} = \mathbf{Y}_{pn} L_n^\top$  and  $\widehat{\mathbf{Z}}_{pn} = \mathbf{Z}_{pn} L_n^\top$  are random matrices with  $m$ -dependent columns

$$\sum_{j: k-m < j \leq k} l_{kj} \mathbf{y}_{pj} \quad \text{and} \quad \sum_{j: k-m < j \leq k} l_{kj} \mathbf{z}_{pj}, \quad k = 1, \dots, n,$$

where  $\mathbf{y}_{p1}, \dots, \mathbf{y}_{pn}$  and  $\mathbf{z}_{p1}, \dots, \mathbf{z}_{pn}$  are columns of  $\mathbf{Y}_{pn}$  and  $\mathbf{Z}_{pn}$  correspondingly.

Therefore, we can proceed along the proof of Theorem 3.1 to show that

$$S_n(z) - \mathbb{E} S_n(z) \rightarrow 0 \quad \text{and} \quad s_n(z) - \mathbb{E} s_n(z) \rightarrow 0 \quad \text{a.s.,} \quad n \rightarrow \infty,$$

where

$$S_n(z) = p^{-1} \text{tr} \left( n^{-1} \mathbf{Y}_{pn} L_n^\top L_n \mathbf{Y}_{pn}^\top - z I_p \right)^{-1},$$

$$s_n(z) = p^{-1} \text{tr} \left( n^{-1} \mathbf{Z}_{pn} L_n^\top L_n \mathbf{Z}_{pn}^\top - z I_p \right)^{-1}.$$

Next we will use Lindeberg's method as in the proofs of Theorem 2.2 and 3.1 to prove that  $\mathbb{E}S_n(z) - \mathbb{E}s_n(z) \rightarrow 0$  as  $n \rightarrow \infty$ . Assume w.l.o.g. that  $\{(\mathbf{y}_{pk}, \Sigma_{pk})\}_{k=1}^n$  are independent copies of  $(\mathbf{y}_p, \Sigma_p)$  as well as  $\mathbf{z}_{pk} = \Sigma_{pk}^{1/2} \mathbf{w}_{pk}$  for all  $1 \leq k \leq n$ , where  $\{\mathbf{w}_{pk}\}_{k=1}^n$  are independent standard normal vectors in  $\mathbb{R}^p$  that are also independent of  $\{(\mathbf{y}_{pk}, \Sigma_{pk})\}_{k=1}^n$ .

We have  $|S_n(z) - s_n(z)| \leq \sum_{k=1}^n |I_{kn}|/p$ , where

$$I_{kn} = \text{tr} \left( n^{-1} \mathbf{Y}_{pn}^k L_n^\top L_n (\mathbf{Y}_{pn}^k)^\top - z I_p \right)^{-1} - \text{tr} \left( n^{-1} \mathbf{Y}_{pn}^{k-1} L_n^\top L_n (\mathbf{Y}_{pn}^{k-1})^\top - z I_p \right)^{-1}$$

for  $p \times n$  matrices  $\mathbf{Y}_{pn}^k$  defined as follows:  $\mathbf{Y}_{pn}^0 = \mathbf{Z}_{pn}$ ,  $\mathbf{Y}_{pn}^n = \mathbf{Y}_{pn}$ , and  $\mathbf{Y}_{pn}^k$  is a matrix with columns

$$\mathbf{y}_{pj}, \quad 1 \leq j \leq k, \quad \text{and} \quad \mathbf{z}_{pj}, \quad k < j \leq n,$$

for all  $1 \leq k < n$ .

Let further  $\mathbf{y}_{pj}$  and  $\mathbf{z}_{pj}$  be zero vectors and  $l_{kj} = 0$  whenever  $k, j$  do not belong to the set  $\{1, \dots, n\}$ . Since

$$A L_n^\top L_n A^\top = \sum_{k=1}^n \left( \sum_{j: k-m < j \leq k} l_{kj} a_j \right) \left( \sum_{j: k-m < j \leq k} l_{kj} a_j \right)^\top.$$

for any  $p \times n$  matrix  $A$  with columns  $a_1, \dots, a_n$ , there are symmetric positive semi-



definite  $p \times p$  matrices  $C_{kn}$ ,  $1 \leq k \leq n$ , such that

$$\frac{1}{n} \mathbf{Y}_{pn}^{k-1} L_n^\top L_n (\mathbf{Y}_{pn}^{k-1})^\top = C_{kn} + \frac{1}{n} \sum_{s=k}^{k+m-1} \mathbf{z}_{ps}^k (\mathbf{z}_{ps}^k)^\top,$$

$$\frac{1}{n} \mathbf{Y}_{pn}^k L_n^\top L_n (\mathbf{Y}_{pn}^k)^\top = C_{kn} + \frac{1}{n} \sum_{s=k}^{k+m-1} \mathbf{y}_{ps}^k (\mathbf{y}_{ps}^k)^\top,$$

where

$$\begin{aligned} \mathbf{z}_{ps}^k &= \sum_{j: s-m < j < k} l_{sj} \mathbf{y}_{pj} + l_{sk} \mathbf{z}_{pk} + \sum_{j: k < j \leq s} l_{sj} \mathbf{z}_{pj}, \\ \mathbf{y}_{ps}^k &= \sum_{j: s-m < j < k} l_{sj} \mathbf{y}_{pj} + l_{sk} \mathbf{y}_{pk} + \sum_{j: k < j \leq s} l_{sj} \mathbf{z}_{pj}. \end{aligned}$$

By this definition,  $C_{kn}$  is independent of  $(\mathbf{y}_{pk}, \mathbf{z}_{pk})$  for any fixed  $k = 1, \dots, n$ .

Denote by  $U_{kn}$  and  $V_{kn}$  random  $p \times m$  matrices with columns

$$n^{-1/2} \mathbf{z}_{ps}^k, \quad k \leq s \leq k+m-1, \quad \text{and} \quad n^{-1/2} \mathbf{y}_{ps}^k, \quad k \leq s \leq k+m-1,$$

correspondingly. Then

$$\text{tr}(n^{-1} \mathbf{Y}_{pn}^k L_n^\top L_n (\mathbf{Y}_{pn}^k)^\top - z I_p)^{-1} = \text{tr}(C_{kn} + V_{kn} V_{kn}^\top - z I_p)^{-1},$$

$$\text{tr}(n^{-1} \mathbf{Y}_{pn}^{k-1} L_n^\top L_n (\mathbf{Y}_{pn}^{k-1})^\top - z I_p)^{-1} = \text{tr}(C_{kn} + U_{kn} U_{kn}^\top - z I_p)^{-1}.$$

By the Sherman-Morrison-Woodbury formula,

$$\begin{aligned} \text{tr}(C + U U^\top - z I_p)^{-1} &= \\ &= \text{tr}(C - z I_p)^{-1} - \text{tr}((C - z I_p)^{-1} U (I_q + U^\top (C - z I_p)^{-1} U)^{-1} U^\top (C - z I_p)^{-1}) \\ &= \text{tr}(C - z I_p)^{-1} - \text{tr}(U^\top (C - z I_p)^{-2} U (I_q + U^\top (C - z I_p)^{-1} U)^{-1}) \end{aligned}$$

for any real symmetric  $p \times p$  matrix  $C$  and all real  $p \times q$  matrices  $U$ . Hence, adding and subtracting  $\text{tr}(C_{kn} - zI_p)^{-1}$  to  $I_{kn}$  yield

$$I_{kn} = \text{tr}(U_{kn}^\top A_{kn}^2 U_{kn} (I_m + U_{kn}^\top A_{kn} U_{kn})^{-1}) - \text{tr}(V_{kn}^\top A_{kn}^2 V_{kn} (I_m + V_{kn}^\top A_{kn} V_{kn})^{-1}),$$

where we set  $A_{kn} = A_{kn}(z) = (C_{kn} - zI_p)^{-1}$ ,  $1 \leq k \leq n$ .

**Lemma 5.10.** *Let  $z \in \mathbb{C}^+$ ,  $U$  be a real  $p \times q$  matrix and  $C$  be a real symmetric positive semi-definite  $p \times p$  matrix. Then*

$$|\text{tr}(U^\top A^2 U (I_q + U^\top A U)^{-1})| \leq \frac{q(|z| + \text{Im}(z))}{|z| \text{Im}(z)},$$

where  $A = (C - zI_p)^{-1}$ .

**Lemma 5.11.** *Let  $z \in \mathbb{C}^+$ ,  $U$  and  $V$  be real  $p \times q$  matrices and  $C$  be a real symmetric positive semi-definite  $p \times p$  matrix. Then*

$$\begin{aligned} & |\text{tr}(U^\top A^2 U (I_q + U^\top A U)^{-1}) - \text{tr}(V^\top A^2 V (I_q + V^\top A V)^{-1})| \leq \\ & \leq K \sum_{j=1}^2 \|U^\top A^j U - V^\top A^j V\| + K \|U^\top A U - V^\top A V\| \|U^\top A A^* U - V^\top A A^* V\|^{1/2}, \end{aligned}$$

where  $K = K(q, z) = q(|z| + 1)^{3/2} / |\text{Im}(z)|^2$ ,  $A = (C - zI_p)^{-1}$  and  $A^* = (C - \bar{z}I_p)^{-1}$ .

In view of Lemma 5.10, we finish the proof if we show that

$$\frac{1}{p} \sum_{k=1}^n \mathbb{E} |I_{kn}| = \frac{1}{p} \sum_{k=1}^n \mathbb{E} (|I_{kn}| \wedge a) \rightarrow 0, \quad n \rightarrow \infty, \quad (16)$$

for  $a = 2m(|z| + v)/(v|z|)$  and  $v = \text{Im}(z) > 0$ . Fix any  $\varepsilon > 0$  and define

$$D_{kn} = \bigcap_{j=1}^2 \{ \|U_{kn}^\top A_{kn}^j U_{kn} - V_{kn}^\top A_{kn}^j V_{kn}\| \leq \varepsilon \} \cap \{ \|U_{kn}^\top A_{kn} A_{kn}^* U_{kn} - V_{kn}^\top A_{kn} A_{kn}^* V_{kn}\| \leq \varepsilon \}.$$

By Lemma 5.11,

$$\begin{aligned} \frac{1}{p} \sum_{k=1}^n \mathbb{E}(|I_{kn}| \wedge a) &= \frac{1}{p} \sum_{k=1}^n \mathbb{E}(|I_{kn}| \wedge a) I(D_{kn}) + \frac{1}{p} \sum_{k=1}^n \mathbb{E}(|I_{kn}| \wedge a) I(\overline{D}_{kn}) \\ &\leq \frac{1}{p} \sum_{k=1}^n \mathbb{E}|I_{kn}| I(D_{kn}) + \frac{a}{p} \sum_{k=1}^n \mathbb{P}(\overline{D}_{kn}) \\ &\leq \frac{n}{p} K(m, z)(2\varepsilon + \varepsilon^{3/2}) + \frac{a}{p} \sum_{k=1}^n \mathbb{P}(\overline{D}_{kn}). \end{aligned}$$

Since  $n/p \rightarrow 1/y > 0$ , to prove (16) we need to show that, for any fixed  $\varepsilon > 0$ ,

$$\frac{1}{p} \sum_{k=1}^n \mathbb{P}(\overline{D}_{kn}) \rightarrow 0.$$

Due to Lemma 5.2, it is clear that  $\|A_{kn}\| \leq M$ ,  $\|A_{kn}^2\| \leq M$  and  $\|A_{kn} A_{kn}^*\| \leq M$  a.s. for  $M = \max\{v^{-1}, v^{-2}\}$ . Moreover,  $A_{kn}$ ,  $A_{kn}^2$  and  $A_{kn} A_{kn}^* = ((C_{kn} - uI_p)^2 + v^2 I_p)^{-1}$  are symmetric matrices. As a result,

$$\mathbb{P}(\overline{D}_{kn}) \leq 3 \sup_{B_{pk}} \mathbb{P}(\|U_{kn}^\top B_{pk} U_{kn} - V_{kn}^\top B_{pk} V_{kn}\| > \varepsilon),$$

where the supremum is taken over all complex symmetric  $p \times p$  random matrices  $B_{pk}$  such that  $\|B_{pk}\| \leq M$  a.s. and  $B_{pk}$  (with  $(\mathbf{y}_{pj}, \mathbf{z}_{pj})$ ,  $j \neq k$ ) is independent of  $(\mathbf{y}_{pk}, \mathbf{z}_{pk})$ .

Recalling the definitions of  $U_{kn}$  and  $V_{kn}$ , we write  $U_{kn} = \widehat{U}_{kn} L_{kn}^\top$  and  $V_{kn} = \widehat{V}_{kn} L_{kn}^\top$ ,

where

$$L_{kn} = \begin{pmatrix} l_{k,k-m+1} & \dots & l_{kk} & 0 & \dots & 0 \\ 0 & l_{k+1,k-m+1} & \dots & l_{k+1,k+1} & 0 & \dots \\ & & & \dots & & \\ \dots & 0 & l_{k+m-1,k} & \dots & & l_{k+m-1,k+m-1} \end{pmatrix}$$

is real  $m \times (2m - 1)$  matrix,  $\widehat{U}_{kn}$  is a  $p \times (2m - 1)$  matrix with columns

$$n^{-1/2} \mathbf{y}_{pj}, \quad k - m + 1 \leq j < k, \quad \text{and} \quad n^{-1/2} \mathbf{z}_{pj}, \quad k \leq j \leq k + m - 1,$$

and  $\widehat{V}_{kn}$  is a  $p \times (2m - 1)$  matrix with columns

$$n^{-1/2} \mathbf{y}_{pj}, \quad k - m + 1 \leq j \leq k, \quad \text{and} \quad n^{-1/2} \mathbf{z}_{pj}, \quad k < j \leq k + m - 1.$$

By the assumption of Theorem 3.3, entries of  $L_n$  are uniformly bounded over  $n$ . In addition,  $L_{kn}$ ,  $1 \leq k \leq n$ , is a submatrix of  $L_n$  which size does not depend on  $n$  and is equal to  $m \times (2m - 1)$ . Therefore,

$$S = \sup\{\|L_{kn}\| \|L_{kn}^\top\| : n \geq 1, k = 1, \dots, n\} < \infty.$$

Hence,

$$\begin{aligned} \|U_{kn}^\top B_{pk} U_{kn} - V_{kn}^\top B_{pk} V_{kn}\| &= \|L_{kn}(\widehat{U}_{kn}^\top B_{pk} \widehat{U}_{kn} - \widehat{V}_{kn}^\top B_{pk} \widehat{V}_{kn})L_{kn}^\top\| \\ &\leq S \|\widehat{U}_{kn}^\top B_{pk} \widehat{U}_{kn} - \widehat{V}_{kn}^\top B_{pk} \widehat{V}_{kn}\|. \end{aligned}$$

For any  $p \times p$  matrix  $A$  and any  $p \times (2m-1)$  matrix  $U$  with columns  $u_1, \dots, u_{2m-1}$ ,

$$U^\top AU = (u_i^\top Au_j)_{i,j=1}^{2m-1}.$$

Hence, applying the bound

$$\|Q\|^2 \leq \|Q^*Q\| \leq \text{tr}(Q^*Q) = \sum_{i,j=1}^d |q_{ij}|^2$$

valid for any complex  $d \times d$  matrix  $Q = (q_{ij})_{i,j=1}^d$ , we get

$$\begin{aligned} \|\widehat{U}_{kn}^\top B_{pk} \widehat{U}_{kn} - \widehat{V}_{kn}^\top B_{pk} \widehat{V}_{kn}\| &\leq (J_{kn} + |\mathbf{y}_{pk}^\top B_{pk} \mathbf{y}_{pk} - \mathbf{z}_{pk}^\top B_{pk} \mathbf{z}_{pk}|^2/n^2)^{1/2} \\ &\leq \sqrt{J_{kn}} + |\mathbf{y}_{pk}^\top B_{pk} \mathbf{y}_{pk} - \mathbf{z}_{pk}^\top B_{pk} \mathbf{z}_{pk}|/n \end{aligned}$$

where

$$J_{kn} = \frac{2}{n^2} \sum_{i=k-m+1}^{k-1} |\mathbf{y}_{pi}^\top B_{pk} \mathbf{y}_{pk} - \mathbf{y}_{pi}^\top B_{pk} \mathbf{z}_{pk}|^2 + \frac{2}{n^2} \sum_{i=k+1}^{k+m-1} |\mathbf{z}_{pi}^\top B_{pk} \mathbf{y}_{pk} - \mathbf{z}_{pi}^\top B_{pk} \mathbf{z}_{pk}|^2.$$

Gathering together these bounds,

$$\begin{aligned} \mathbb{P}(\|U_{kn}^\top B_{pk} U_{kn} - V_{kn}^\top B_{pk} V_{kn}\| > \varepsilon) &\leq \mathbb{P}(\|\widehat{U}_{kn}^\top B_{pk} \widehat{U}_{kn} - \widehat{V}_{kn}^\top B_{pk} \widehat{V}_{kn}\| > \varepsilon/S) \\ &\leq \mathbb{P}(J_{kn}^{1/2} > \varepsilon/(3S)) + \mathbb{P}(|\mathbf{z}_{pk}^\top B_{pk} \mathbf{z}_{pk} - \text{tr}(\Sigma_{pk} B_{pk})| > \varepsilon/(3S)) \\ &\quad + \mathbb{P}(|\mathbf{y}_{pk}^\top B_{pk} \mathbf{y}_{pk} - \text{tr}(\Sigma_{pk} B_{pk})| > \varepsilon/(3S)) \\ &\leq \frac{9S^2}{\varepsilon^2} \mathbb{E} J_{kn} + \sup_{A_p} \mathbb{P}(|\mathbf{z}_p^\top A_p \mathbf{z}_p - \text{tr}(\Sigma_p A_p)| > \varepsilon/(3S)) \\ &\quad + \sup_{A_p} \mathbb{P}(|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)| > \varepsilon/(3S)), \end{aligned}$$

where the supremum is taken over all complex  $p \times p$  matrices  $A_p$  with  $\|A_p\| \leq M$ .

By (A1) and Lemma 5.3,

$$\sup_{A_p} \mathbb{P}(|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)| > \varepsilon/(3S)) = o(1), \quad p \rightarrow \infty.$$

By (A2) and Proposition 2.1,

$$\sup_{A_p} \mathbb{P}(|\mathbf{z}_p^\top A_p \mathbf{z}_p - \text{tr}(\Sigma_p A_p)| > \varepsilon/(3S)) = o(1).$$

All these bounds guarantee that (for fixed  $\varepsilon > 0$  and  $D_{kn} = D_{kn}(\varepsilon)$ )

$$\frac{1}{p} \sum_{k=1}^n \mathbb{P}(\overline{D}_{kn}) \rightarrow 0$$

whenever  $m$  is fixed,  $n \rightarrow \infty$ ,  $p/n \rightarrow y > 0$ , and

$$\frac{1}{p} \sum_{k=1}^n \sup_{B_{pk}} \mathbb{E} J_{kn} \rightarrow 0.$$

Thus we need to verify the last relation in order to finish the proof of the theorem.

By the Cauchy inequality,

$$\begin{aligned} \mathbb{E} J_{kn} &\leq \frac{4}{n^2} \sum_{i=k-m+1}^{k-1} (\mathbb{E} |\mathbf{y}_{pi}^\top B_{pk} \mathbf{y}_{pk}|^2 + \mathbb{E} |\mathbf{y}_{pi}^\top B_{pk} \mathbf{z}_{pk}|^2) \\ &\quad + \frac{4}{n^2} \sum_{i=k+1}^{k+m-1} (\mathbb{E} |\mathbf{z}_{pi}^\top B_{pk} \mathbf{y}_{pk}|^2 + \mathbb{E} |\mathbf{z}_{pi}^\top B_{pk} \mathbf{z}_{pk}|^2). \end{aligned}$$

Using independence of  $(\mathbf{y}_{pk}, \mathbf{z}_{pk})$  and  $(\mathbf{y}_{pi}, \mathbf{z}_{pi}, B_{pk})$ ,  $i \neq k$ , inequality  $\|B_{pk}\| \leq M$  a.s.

and arguing as in the end of the proof of Theorem 3.1 (see (14) and (15)),

$$\begin{aligned}
\mathbb{E}|\mathbf{y}_{pi}^\top B_{pk} \mathbf{y}_{pk}|^2 &\leq M^2 \|\mathbb{E} \mathbf{y}_{pk} \mathbf{y}_{pk}^\top\| \text{tr}(\mathbb{E} \mathbf{y}_{pi} \mathbf{y}_{pi}^\top) = M^2 \|\mathbb{E} \mathbf{y}_p \mathbf{y}_p^\top\| \text{tr}(\mathbb{E} \mathbf{y}_p \mathbf{y}_p^\top), \\
\mathbb{E}|\mathbf{z}_{pi}^\top B_{pk} \mathbf{y}_{pk}|^2 &\leq M^2 \|\mathbb{E} \mathbf{y}_{pk} \mathbf{y}_{pk}^\top\| \text{tr}(\mathbb{E} \Sigma_{pi}) = M^2 \|\mathbb{E} \mathbf{y}_p \mathbf{y}_p^\top\| \text{tr}(\mathbb{E} \Sigma_p), \\
\mathbb{E}|\mathbf{y}_{pi}^\top B_{pk} \mathbf{z}_{pk}|^2 &\leq M^2 \|\mathbb{E} \Sigma_{pk}\| \text{tr}(\mathbb{E} \mathbf{y}_{pi} \mathbf{y}_{pi}^\top) = M^2 \|\mathbb{E} \Sigma_p\| \text{tr}(\mathbb{E} \mathbf{y}_p \mathbf{y}_p^\top), \\
\mathbb{E}|\mathbf{z}_{pi}^\top B_{pk} \mathbf{z}_{pk}|^2 &\leq M^2 \|\mathbb{E} \Sigma_{pk}\| \text{tr}(\mathbb{E} \Sigma_{pi}) = M^2 \|\mathbb{E} \Sigma_p\| \text{tr}(\mathbb{E} \Sigma_p).
\end{aligned}$$

It now follows from (A6) and (31) that

$$\frac{1}{p} \sum_{k=1}^n \sup_{B_{pk}} \mathbb{E} J_{kn} \leq \frac{4M^2(m-1)}{pn} (\|\mathbb{E} \Sigma_p\| + \|\mathbb{E} \mathbf{y}_p \mathbf{y}_p^\top\|) \text{tr}(\mathbb{E} \Sigma_p + \mathbb{E} \mathbf{y}_p \mathbf{y}_p^\top) = o(1).$$

This finishes the proof of the theorem. Q.e.d.

**Proof of Theorem 4.1.** We will prove the first inequality by the same arguments as in the proof of Theorem 3 in [6]. Write  $a = (a_1, \dots, a_p)$ . By Lemma 1 in [6],

$$|(\mathbf{x}_p^\top a)^4 - 24T| \leq C_0 \sum_{j=0}^2 |\mathbf{x}_p^\top a|^j |S|^{4-j},$$

where  $C_0 > 0$  is a universal constant,

$$T = \sum_{i < j < k < l} a_i a_j a_k a_l X_i X_j X_k X_l, \quad S = \left( \sum_{i=1}^p a_i^2 X_i^2 \right)^{1/2},$$

hereinafter  $i, j, k, l$  are any numbers in  $\{1, \dots, p\}$ . Hence, by Minkowski's inequality,

$$\begin{aligned}
\mathbb{E}(\mathbf{x}_p^\top a)^4 &\leq 24\mathbb{E}T + C_0 \sum_{j=0}^2 \mathbb{E}|\mathbf{x}_p^\top a|^j |S|^{4-j} \\
&\leq 24\mathbb{E}T + C_0 \sum_{j=0}^2 [\mathbb{E}(\mathbf{x}_p^\top a)^4]^{j/4} (\mathbb{E}S^4)^{1-j/4}.
\end{aligned}$$

By the Cauchy-Schwartz and (3),

$$|\mathbb{E}T| \leq \sum_{i < j < k < l} |a_i a_j a_k a_l| \min\{\varphi_{j-i}, \varphi_{k-j}, \varphi_{l-k}\} \leq \sqrt{J_1 J_2}$$

with

$$J_1 = \sum_{i < j < k < l} a_i^2 a_j^2 \min\{\varphi_{k-j}, \varphi_{l-k}\}, \quad J_2 = \sum_{i < j < k < l} a_k^2 a_l^2 \min\{\varphi_{j-i}, \varphi_{k-j}\}.$$

In addition,

$$\begin{aligned} J_1 &\leq \sum_{i < j} a_i^2 a_j^2 \sum_{k: k > j} \left( (k-j)\varphi_{k-j} + \sum_{l: l-k > k-j} \varphi_{l-k} \right) \leq \\ &\leq \frac{\|a\|^4}{2} \Phi_1 + \frac{\|a\|^4}{2} \sum_{q=1}^{\infty} \sum_{r=q+1}^{\infty} \varphi_r = \\ &= \frac{\|a\|^4}{2} \Phi_1 + \frac{\|a\|^4}{2} \sum_{r=2}^{\infty} (r-1)\varphi_r \leq \Phi_1 \|a\|^4. \end{aligned}$$

Similar arguments yield

$$J_2 \leq \sum_{k < l} a_k^2 a_l^2 \sum_{j: j < k} \left( (k-j)\varphi_{k-j} + \sum_{i: j-i > k-j} \varphi_{j-i} \right) \leq \Phi_1 \|a\|^4.$$

Let us also note that

$$\mathbb{E}|S|^4 = \sum_{i,j=1}^p a_i^2 a_j^2 \mathbb{E}X_i^2 X_j^2 \leq \Phi_0 \|a\|^4.$$

Combining the above estimates, we infer that

$$\mathbb{E}(\mathbf{x}_p^\top a)^4 \leq 24(\Phi_0 + \Phi_1) \|a\|^4 + C_0 \sum_{j=0}^2 [\mathbb{E}(\mathbf{x}_p^\top a)^4]^{j/4} ((\Phi_0 + \Phi_1) \|a\|^4)^{1-j/4}.$$



Put  $R = (\mathbb{E}(\mathbf{x}_p^\top a)^4)^{1/4}(\Phi_0 + \Phi_1)^{-1/4}/\|a\|$ . Then

$$R^4 \leq 24 + C_0 + C_0 R + C_0 R^2.$$

Hence,  $R \leq R_0$ , where  $R_0 > 0$  is the largest root of the equation

$$x^4 = 24 + C_0 + C_0 x + C_0 x^2.$$

Finally we conclude that  $\mathbb{E}|\mathbf{x}^\top a|^4 \leq R_0^4(\Phi_0 + \Phi_1)\|a\|^4$ .

We now verify the second inequality. Let  $A = (a_{ij})_{i,j=1}^p$  and  $a_{ii} = 0$ ,  $1 \leq i \leq p$ .

Since  $\mathbf{x}_p^\top A \mathbf{x}_p = \mathbf{x}_p^\top B \mathbf{x}_p$  and

$$\text{tr}(BB^\top) = \sum_{i,j=1}^p \left( \frac{a_{ij} + a_{ji}}{2} \right)^2 \leq \sum_{i,j=1}^p \frac{a_{ij}^2 + a_{ji}^2}{2} = \sum_{i,j=1}^p a_{ij}^2 = \text{tr}(AA^\top) \quad (17)$$

for  $B = (A^\top + A)/2$ , we may assume that  $A = A^\top$ . Then

$$\begin{aligned} \mathbb{E}|\mathbf{x}_p^\top A \mathbf{x}_p|^2 &= 4\mathbb{E} \left| \sum_{i=1}^{p-1} X_i \sum_{k=i+1}^p a_{ik} X_k \right|^2 \\ &= 4 \sum_{i=1}^{p-1} \mathbb{E} X_i^2 \left| \sum_{k=i+1}^p a_{ik} X_k \right|^2 + 8 \sum_{i < j} \mathbb{E} X_i X_j \left( \sum_{k=i+1}^p a_{ik} X_k \right) \left( \sum_{k=j+1}^p a_{jk} X_k \right) \\ &= 4I_1 + 8I_2 + 8I_3 + 8I_4, \end{aligned}$$

where

$$\begin{aligned}
I_1 &= \sum_{i=1}^{p-1} \mathbb{E} X_i^2 \left| \sum_{k=i+1}^p a_{ik} X_k \right|^2, \\
I_2 &= \sum_{i < j} \mathbb{E} X_i \left( \sum_{k=i+1}^{j-1} a_{ik} X_k \right) X_j \left( \sum_{k=j+1}^p a_{jk} X_k \right), \\
I_3 &= \sum_{i < j} a_{ij} \mathbb{E} X_i X_j^2 \left( \sum_{k=j+1}^p a_{jk} X_k \right), \\
I_4 &= \sum_{i < j} \mathbb{E} X_i X_j \left( \sum_{k=j+1}^p a_{ik} X_k \right) \left( \sum_{k=j+1}^p a_{jk} X_k \right)
\end{aligned}$$

and sums over the empty set are zeros.

*Control of  $I_1$ .* By the Cauchy-Schwartz inequality and the first part of Theorem 4.1,

$$\begin{aligned}
I_1 &\leq \sum_{i=1}^{p-1} \sqrt{\mathbb{E} X_i^4} \left( \mathbb{E} \left| \sum_{k=i+1}^p a_{ik} X_k \right|^4 \right)^{1/2} \\
&\leq C(\Phi_0 + \Phi_1) \sum_{i=1}^{p-1} \sum_{k=i+1}^p a_{ik}^2 \\
&= C(\Phi_0 + \Phi_1) \frac{\text{tr}(A^2)}{2}
\end{aligned}$$

*Control of  $I_2$ .* By the Cauchy-Schwartz inequality and (3),

$$\begin{aligned}
I_2 &\leq \sum_{i < k < j < l} |a_{ik} a_{jl}| |\mathbb{E} X_i X_k X_j X_l| \\
&\leq \sum_{i < k < j < l} |a_{ik} a_{jl}| \min\{\varphi_{k-i}, \varphi_{j-k}, \varphi_{l-j}\} \\
&\leq \sqrt{I_5 I_6},
\end{aligned}$$

where

$$I_5 = \sum_{i < k < j < l} a_{ik}^2 \min\{\varphi_{j-k}, \varphi_{l-j}\}, \quad I_6 = \sum_{i < k < j < l} a_{jl}^2 \min\{\varphi_{k-i}, \varphi_{j-k}\}.$$

Additionally,

$$\begin{aligned} I_5 &\leq \sum_{i < k} a_{ik}^2 \sum_{j: j > k} \left( (j-k)\varphi_{j-k} + \sum_{l: l-j > j-k} \varphi_{l-j} \right) \\ &\leq \frac{\text{tr}(A^2)}{2} \Phi_1 + \frac{\text{tr}(A^2)}{2} \sum_{q=1}^{\infty} \sum_{r=q+1}^{\infty} \varphi_r = \\ &= \frac{\text{tr}(A^2)}{2} \Phi_1 + \frac{\text{tr}(A^2)}{2} \sum_{r=2}^{\infty} (r-1)\varphi_r \leq \text{tr}(A^2)\Phi_1. \end{aligned}$$

We similarly derive that

$$I_6 \leq \sum_{j < l} a_{jl}^2 \sum_{k: k < j} \left( (j-k)\varphi_{j-k} + \sum_{i: k-i > j-k} \varphi_{k-i} \right) \leq \text{tr}(A^2)\Phi_1.$$

Hence,  $I_2 \leq \text{tr}(A^2)\Phi_1$ .

*Control of  $I_3$ .* By the Cauchy-Schwartz inequality and the first part of Theorem 4.1,

$$\begin{aligned} I_3 &= \sum_{j=2}^{p-1} \mathbb{E} X_j^2 \left( \sum_{i=1}^{j-1} a_{ij} X_i \right) \left( \sum_{k=j+1}^p a_{jk} X_k \right) \\ &\leq \sum_{j=2}^{p-1} \sqrt{\mathbb{E} X_j^4} \left[ \mathbb{E} \left( \sum_{i=1}^{j-1} a_{ij} X_i \right)^4 \mathbb{E} \left( \sum_{k=j+1}^p a_{jk} X_k \right)^4 \right]^{1/4} \\ &\leq \sqrt{C(\Phi_0 + \Phi_1)} I_7 I_8, \end{aligned}$$

where

$$I_7 = \sum_{j=2}^{p-1} \left[ \mathbb{E} \left( \sum_{i=1}^{j-1} a_{ij} X_i \right)^4 \right]^{1/2}, \quad I_8 = \sum_{j=2}^{p-1} \left[ \mathbb{E} \left( \sum_{k=j+1}^p a_{jk} X_k \right)^4 \right]^{1/2}.$$

By the first inequality in Theorem 4.1,

$$I_7 \leq K \sum_{j=2}^{p-1} \sum_{i=1}^{j-1} a_{ij}^2 \leq \frac{K \operatorname{tr}(A^2)}{2}, \quad I_8 \leq K \sum_{j=2}^{p-1} \sum_{k=j+1}^p a_{jk}^2 \leq \frac{K \operatorname{tr}(A^2)}{2},$$

where  $K = \sqrt{C(\Phi_0 + \Phi_1)}$ . As a result,  $I_3 \leq C(\Phi_0 + \Phi_1) \operatorname{tr}(A^2)/2$ .

*Control of  $I_4$ .* We have  $I_4 = I_9 + I_{10} + I_{11}$ , where

$$I_9 = \sum_{i < j < k} \mathbb{E}(a_{ik} X_i)(a_{jk} X_j) X_k^2, \quad I_{10} = \sum_{i < j < k < l} a_{ik} a_{jl} \mathbb{E} X_i X_j X_k X_l,$$

$$I_{11} = \sum_{i < j < k < l} a_{il} a_{jk} \mathbb{E} X_i X_j X_k X_l.$$

By the first inequality in Theorem 4.1,

$$\begin{aligned} I_9 &= \frac{1}{2} \sum_{k=3}^p \mathbb{E} \left( \sum_{i=1}^{k-1} a_{ik} X_i \right)^2 X_k^2 - \frac{1}{2} \sum_{k=3}^p \mathbb{E} X_k^2 \sum_{i=1}^{k-1} a_{ik}^2 X_i^2 \\ &\leq \frac{1}{2} \sum_{k=3}^p \left[ \mathbb{E} \left( \sum_{i=1}^{k-1} a_{ik} X_i \right)^4 \right]^{1/2} \sqrt{\mathbb{E} X_k^4} \\ &\leq C(\Phi_0 + \Phi_1) \sum_{k=3}^p \sum_{i=1}^{k-1} \frac{a_{ik}^2}{2} \\ &\leq C(\Phi_0 + \Phi_1) \frac{\operatorname{tr}(A^2)}{4}. \end{aligned}$$

We will estimate  $I_{10}$  and  $I_{11}$  in the same way as  $I_2$ . It follows from the Cauchy-Schwartz inequality that  $I_{10} \leq \sqrt{I_{12} I_{13}}$  and  $I_{11} \leq \sqrt{I_{14} I_{15}}$  with

$$I_{12} = \sum_{i < j < k < l} a_{ik}^2 \min\{\varphi_{j-i}, \varphi_{l-k}\}, \quad I_{13} = \sum_{i < j < k < l} a_{jl}^2 \min\{\varphi_{j-i}, \varphi_{l-k}\},$$

$$I_{14} = \sum_{i < j < k < l} a_{il}^2 \min\{\varphi_{j-i}, \varphi_{l-k}\}, \quad I_{15} = \sum_{i < j < k < l} a_{jk}^2 \min\{\varphi_{j-i}, \varphi_{l-k}\},$$

As previously, we have

$$\begin{aligned}
I_{12} &\leq \sum_{i < k} a_{ik}^2 \sum_{j: i < j < k} \left( (j-i)\varphi_{j-i} + \sum_{l: l-k > j-i} \varphi_{l-k} \right) \leq \text{tr}(A^2)\Phi_1, \\
I_{13} &\leq \sum_{j < l} a_{jl}^2 \sum_{k: j < k < l} \left( (l-k)\varphi_{l-k} + \sum_{i: j-i > l-k} \varphi_{j-i} \right) \leq \text{tr}(A^2)\Phi_1, \\
I_{14} &\leq \sum_{i < l} a_{il}^2 \sum_{k: i < k < l} \left( (l-k)\varphi_{l-k} + \sum_{j: j-i > l-k, j < k} \varphi_{j-i} \right) \leq \text{tr}(A^2)\Phi_1, \\
I_{15} &\leq \sum_{j < k} a_{jk}^2 \sum_{i: i < j} \left( (j-i)\varphi_{j-i} + \sum_{l: l-k > j-i} \varphi_{l-k} \right) \leq \text{tr}(A^2)\Phi_1.
\end{aligned}$$

Thus,  $I_4 \leq C(\Phi_0 + \Phi_1)\text{tr}(A^2)/4 + 2\text{tr}(A^2)\Phi_1$ .

Combining all above estimates, we get

$$\mathbb{E}|\mathbf{x}_p^\top A \mathbf{x}_p|^2 \leq (8C(\Phi_0 + \Phi_1) + 24\Phi_1)\text{tr}(A^2).$$

Q.e.d.

**Proof of Theorem 4.3.** Let  $A = (a_{ij})_{i,j=1}^p$  and  $D$  be the  $p \times p$  diagonal matrix with diagonal entries  $a_{11}, \dots, a_{pp}$ . By Theorem 4.1,

$$\mathbb{E}|\mathbf{x}_p^\top (A - D) \mathbf{x}_p|^2 \leq C(\Phi_0 + \Phi_1)\text{tr}((A - D)(A - D)^\top).$$

In addition,  $\text{tr}(\Sigma_p A) = \text{tr}(\Sigma_p D)$  for the diagonal matrix  $\Sigma_p$  with diagonal entries  $\mathbb{E}X_1^2, \dots, \mathbb{E}X_p^2$  as well as

$$\mathbb{E}|\mathbf{x}_p^\top A \mathbf{x}_p - \text{tr}(\Sigma_p A)|^2 \leq 2\mathbb{E}|\mathbf{x}_p^\top D \mathbf{x}_p - \text{tr}(\Sigma_p D)|^2 + 2\mathbb{E}|\mathbf{x}_p^\top (A - D) \mathbf{x}_p|^2.$$

Since

$$\text{tr}(AA^\top) = \text{tr}((A - D)(A - D)^\top) + \text{tr}(D^2),$$

we only need to bound  $\mathbb{E}|\mathbf{x}_p^\top D \mathbf{x}_p - \text{tr}(\Sigma_p D)|^2$  from above by  $\text{tr}(D^2)$  up to a constant factor. Write  $D = D_1 - D_2$ , where  $D_i$  are diagonal matrices with non-negative diagonal elements. Since

$$\mathbb{E}|\mathbf{x}_p^\top D \mathbf{x}_p - \text{tr}(\Sigma_p D)|^2 \leq 2 \sum_{i=1}^2 \mathbb{E}|\mathbf{x}_p^\top D_i \mathbf{x}_p - \text{tr}(\Sigma_p D_i)|^2,$$

we may assume w.l.o.g. that diagonal elements of  $D$  are non-negative.

We see that

$$\begin{aligned} \mathbb{E}|\mathbf{x}_p^\top D \mathbf{x}_p - \text{tr}(\Sigma_p D)|^2 &= \mathbb{E} \left| \sum_{i=1}^p a_{ii} (X_i^2 - \mathbb{E} X_i^2) \right|^2 \\ &= \sum_{i=1}^p a_{ii}^2 \text{Var}(X_i^2) + \sum_{i \neq j} a_{ii} a_{jj} \text{Cov}(X_i^2, X_j^2) \\ &\leq \Phi_0 \text{tr}(D^2) + \sum_{i \neq j} \frac{a_{ii}^2 + a_{jj}^2}{2} \phi_{|i-j|} \\ &\leq \Phi_0 \text{tr}(D^2) + \sum_{i=1}^p a_{ii}^2 \sum_{j: j \neq i} \phi_{|i-j|} \\ &\leq 2 \text{tr}(D^2) \left( \Phi_0 + \sum_{k=1}^{\infty} \phi_k \right) = 2(\Phi_0 + \Phi_2) \text{tr}(D^2). \end{aligned}$$

Combining the above bounds, we get the desired inequality. Q.e.d.

**Proof of Theorem 4.5.** As in the proof of Theorem 4.1, we may assume that  $A$  is symmetric (see (17)). Note that

$$V_k = \left( \sum_{i=1}^{k-1} a_{ik} X_i \right) X_k, \quad 2 \leq k \leq p,$$

is a martingale difference sequence and

$$\mathbf{x}_p^\top A \mathbf{x}_p = 2 \sum_{1 \leq i < k \leq p} a_{ik} X_i X_k = 2 \sum_{k=2}^p V_k,$$

we infer

$$\begin{aligned} \mathbb{E} \left| \sum_{k=2}^p V_k \right|^2 &= \sum_{k=2}^p \mathbb{E} V_k^2 = \sum_{k=2}^p \mathbb{E} \left( \sum_{i=1}^{k-1} a_{ik} X_i \right)^2 \mathbb{E}(X_k^2 | X_1, \dots, X_{k-1}) \leq \\ &\leq M \sum_{k=2}^p \mathbb{E} \left( \sum_{i=1}^{k-1} a_{ik} X_i \right)^2 = M^2 \sum_{k=2}^p \sum_{i=1}^{k-1} a_{ik}^2 \leq \frac{M^2 \text{tr}(A^2)}{2}. \end{aligned}$$

The latter implies the desired bound. Q.e.d.

**Proof of Theorem 4.6.** Write  $A = (a_{ik})_{i,k=1}^p$ . As in the proof of Theorem 4.1, we may assume that  $A$  is symmetric (see (17)). We have

$$\mathbf{x}_p^\top A \mathbf{x}_p = \sum_{k=1}^p a_{kk} X_k^2 + \sum_{i \neq k} a_{ik} X_i X_k$$

and, by the Cauchy-Schwartz inequality,

$$\mathbb{E} |\mathbf{x}_p^\top A \mathbf{x}_p - \text{tr}(A)| \leq \mathbb{E} \left| \sum_{k=1}^p a_{kk} (X_k^2 - 1) \right| + \left( \mathbb{E} \left| \sum_{i \neq k} a_{ik} X_i X_k \right|^2 \right)^{1/2}.$$

By the Burkholder-Davis-Gundy inequality,

$$\mathbb{E} \left| \sum_{k=1}^p a_{kk} (X_k^2 - 1) \right| \leq C \mathbb{E} \left| \sum_{k=1}^p a_{kk}^2 (X_k^2 - 1)^2 \right|^{1/2},$$

where  $C$  is a universal constant. Using inequality

$$\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}, \quad x, y \geq 0, \quad (18)$$

we derive that

$$\mathbb{E} \left| \sum_{k=1}^p a_{kk}^2 (X_k^2 - 1)^2 \right|^{1/2} \leq I_1 + I_2,$$

where

$$I_1 = \mathbb{E} \left| \sum_{k=1}^p a_{kk}^2 (X_k^2 - 1)^2 I(|X_k^2 - 1| \leq b^2) \right|^{1/2},$$

$$I_2 = \mathbb{E} \left| \sum_{k=1}^p a_{kk}^2 (X_k^2 - 1)^2 I(|X_k^2 - 1| > b^2) \right|^{1/2}.$$

By Jensen's inequality,

$$I_1 \leq \left| \sum_{k=1}^p a_{kk}^2 \mathbb{E} (X_k^2 - 1)^2 I(|X_k^2 - 1| \leq b^2) \right|^{1/2} \leq b \sqrt{2 \operatorname{tr}(AA^\top)}.$$

Here we also used that

$$\mathbb{E} (X_k^2 - 1)^2 I(|X_k^2 - 1| \leq b^2) \leq b^2 \mathbb{E} |X_k^2 - 1| \leq 2b^2, \quad 1 \leq k \leq n.$$

In addition, it follows from (18) that

$$I_2 \leq \sum_{k=1}^p |a_{kk}| \mathbb{E} |X_k^2 - 1| I(|X_k^2 - 1| > b^2).$$

By Theorem 4.5,

$$\mathbb{E} \left| \sum_{i \neq k} a_{ik} X_i X_k \right|^2 \leq 2 \operatorname{tr}(AA^\top).$$

Combining the above estimates, we get the desired result. Q.e.d.

**Proof of Corollary 4.8.** If  $\mathbf{y}_p \in \mathcal{X}_p$ , then  $\Gamma_{pn} \mathbf{x}_n \rightarrow \mathbf{y}_p$  in probability and in mean square as  $n \rightarrow \infty$  for some  $p \times n$  matrices  $\Gamma_{pn}$  and  $\mathbf{x}_n = (X_1, \dots, X_n)$ .



Since  $(X_k)_{k \geq 1}$  is an orthonormal sequence, we have

$$\Gamma_{pn}\Gamma_{pn}^\top = \mathbb{E}(\Gamma_{pn}\mathbf{x}_n)(\Gamma_{pn}\mathbf{x}_n)^\top \rightarrow \mathbb{E}\mathbf{y}_p\mathbf{y}_p^\top = \Sigma_p,$$

$$\mathbf{x}_n^\top(\Gamma_{pn}^\top A_p \Gamma_{pn})\mathbf{x}_n = (\Gamma_{pn}\mathbf{x}_n)^\top A_p \Gamma_{pn}\mathbf{x}_n \xrightarrow{p} \mathbf{y}_p^\top A_p \mathbf{y}_p$$

and  $\text{tr}(\Gamma_{pn}^\top A_p \Gamma_{pn}) = \text{tr}(\Gamma_{pn}\Gamma_{pn}^\top A_p) \rightarrow \text{tr}(\Sigma_p A_p)$  as  $n \rightarrow \infty$ . We need the following version of Fatou's lemma:

$$\text{If } \xi_n \xrightarrow{p} \xi, \text{ then } \mathbb{E}|\xi| \leq \liminf_{n \rightarrow \infty} \mathbb{E}|\xi_n|. \quad (19)$$

By this lemma and Theorem 4.3,

$$\begin{aligned} \mathbb{E}|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)|^2 &\leq \liminf_{n \rightarrow \infty} \mathbb{E}|\mathbf{x}_n^\top(\Gamma_{pn}^\top A_p \Gamma_{pn})\mathbf{x}_n - \text{tr}(\Gamma_{pn}^\top A_p \Gamma_{pn})|^2 \\ &\leq \liminf_{n \rightarrow \infty} C(\Phi_0 + \Phi_1 + \Phi_2) \text{tr}(\Gamma_{pn}^\top A_p \Gamma_{pn} \Gamma_{pn}^\top A_p \Gamma_{pn}) \end{aligned}$$

Note that

$$\text{tr}(\Gamma_{pn}^\top A_p \Gamma_{pn} \Gamma_{pn}^\top A_p \Gamma_{pn}) = \text{tr}(\Gamma_{pn}\Gamma_{pn}^\top A_p \Gamma_{pn}\Gamma_{pn}^\top A_p) \rightarrow \text{tr}(\Sigma_p A_p \Sigma_p A_p).$$

Let  $A_p^{1/2}$  be the principal square root of  $A_p$ . Then

$$R := \text{tr}(\Sigma_p A_p \Sigma_p A_p) = \text{tr}(A_p^{1/2} \Sigma_p A_p \Sigma_p A_p^{1/2}).$$

Since  $\|A_p\|I_p - A_p$  is positive semi-definite, then  $Q^\top(\|A_p\|I_p - A_p)Q$  is positive semi-definite for any matrix  $Q$ . Taking  $Q = \Sigma_p A_p^{1/2}$ , we get

$$R = \text{tr}(Q^\top A_p Q) \leq \|A_p\| \text{tr}(Q^\top Q) = \|A_p\| \text{tr}(Q Q^\top) = \|A_p\| \text{tr}(\Sigma_p A_p \Sigma_p). \quad (20)$$

Analogously,  $\text{tr}(\Sigma_p A_p \Sigma_p) \leq \|A_p\| \text{tr}(\Sigma_p^2)$ . Hence, we obtain the desired bound.

In particular, we get that

$$\mathbb{E}|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)|/p|^2 = O(\|A_p\|^2 \text{tr}(\Sigma_p^2)/p^2), \quad p \rightarrow \infty.$$

Thus (A1) holds when (A2) holds. Q.e.d.

**Proof of Corollary 4.9.** Arguing as in the proof of Corollary 4.8 and using the same notation, we derive

$$\mathbb{E}|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)| \leq \lim_{n \rightarrow \infty} \mathbb{E}|\mathbf{x}_n^\top (\Gamma_{pn}^\top A_p \Gamma_{pn}) \mathbf{x}_n - \text{tr}(\Gamma_{pn}^\top A_p \Gamma_{pn})|$$

If  $A_p$  is a symmetric positive semi-definite matrix, then  $B_n = \Gamma_{pn}^\top A_p \Gamma_{pn}$  has the same properties. Particularly, diagonal entries of  $B_n$  are non-negative. Theorem 4.6 yields

$$\mathbb{E}|\mathbf{x}_n^\top B_n \mathbf{x}_n - \text{tr}(B_n)| \leq Cb\sqrt{\text{tr}(B_n B_n^\top)} + CL(b)\text{tr}(B_n)$$

for all  $b, n > 1$ , some universal constant  $C > 0$  and

$$L(b) = \sup_{k \geq 1} \mathbb{E}|X_k^2 - 1|I(|X_k^2 - 1| > b^2).$$

As in the proof of Corollary 4.8, we get

$$\lim_{n \rightarrow \infty} \text{tr}(B_n B_n^\top) = \text{tr}(\Sigma_p A_p \Sigma_p A_p) \leq \|A_p\|^2 \text{tr}(\Sigma_p^2),$$

$$\lim_{n \rightarrow \infty} \text{tr}(B_n) = \text{tr}(\Sigma_p A_p) \leq \|A_p\| \text{tr}(\Sigma_p).$$

Thus we have proven that

$$\mathbb{E}|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)| \leq Cb\|A_p\| \sqrt{\text{tr}(\Sigma_p^2)} + CL(b)\|A_p\|\text{tr}(\Sigma_p).$$

Take any sequence of symmetric positive semi-definite  $p \times p$  matrices  $A_p$  with  $\|A_p\| = O(1)$  ( $p = 1, 2, \dots$ ) and assume that  $\text{tr}(\Sigma_p^2) = o(p^2)$  holds and  $\text{tr}(\Sigma_p) = O(p)$ . Then, for any  $b > 1$ ,

$$\mathbb{E}|[\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)]/p| \leq o(1) + L(b)T_p, \quad p \rightarrow \infty,$$

where  $T_p = O(1)$  does not depend on  $b$ . If  $\{X_k^2\}_{k=1}^\infty$  is a uniformly integrable family, then  $L(b) \rightarrow 0$  as  $b \rightarrow \infty$ , since

$$\mathbb{E}|X_k^2 - 1|I(|X_k^2 - 1| > b^2) \leq 2\mathbb{E}X_k^2 I(X_k^2 > b^2 + 1) \leq 2\mathbb{E}X_k^2 I(|X_k| > b) \rightarrow 0$$

uniformly in  $k$  as  $b \rightarrow \infty$ . As a result, we conclude that (A1) holds. Q.e.d.

## A Appendix

**Proof of Proposition 2.1.** We need to show that

$$\mathbb{P}(|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)| > \varepsilon p) \leq \mathbb{E} \min \{16M^2 \text{tr}(\Sigma_p^2)(\varepsilon p)^{-2}, 1\} \quad (21)$$

for any  $\varepsilon, M > 0$  and each complex  $p \times p$  matrix  $A_p$  with  $\|A_p\| \leq M$ . In particular, this inequality implies that (A1) holds when (A2) holds.

By Chebyshev's inequality,

$$\begin{aligned}\mathbb{P}(|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)| > \varepsilon p) &= \mathbb{E}\mathbb{P}(|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)| > \varepsilon p | \Sigma_p) \\ &\leq \mathbb{E} \min \left\{ \mathbb{E}(|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)|^2 | \Sigma_p) (\varepsilon p)^{-2}, 1 \right\}.\end{aligned}$$

Write  $A_p = A_{p1} + iA_{p2}$  for real  $p \times p$  matrices  $A_{pj}$ ,  $j = 1, 2$ . Then

$$|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)|^2 = \sum_{j=1}^2 |\mathbf{y}_p^\top A_{pj} \mathbf{y}_p - \text{tr}(\Sigma_p A_{pj})|^2.$$

We have

$$\mathbf{y}_p^\top A_{pj} \mathbf{y}_p = \mathbf{y}_p^\top A_{pj}^\top \mathbf{y}_p = \mathbf{y}_p^\top C_{pj} \mathbf{y}_p,$$

$$\text{tr}(\Sigma_p A_{pj}) = \text{tr}(A_{pj}^\top \Sigma_p) = \text{tr}(A_{pj}^\top \Sigma_p) = \text{tr}(\Sigma_p A_{pj}^\top) = \text{tr}(\Sigma_p C_{pj}),$$

where  $C_{pj} = (A_{pj}^\top + A_{pj})/2$  and  $j = 1, 2$ . Hence,

$$|\mathbf{y}_p^\top A_{pj} \mathbf{y}_p - \text{tr}(\Sigma_p A_{pj})| = |\mathbf{y}_p^\top C_{pj} \mathbf{y}_p - \text{tr}(\Sigma_p C_{pj})|.$$

Let us show that  $\|C_{pj}\| \leq M$  for  $j = 1, 2$ . Using standard properties of the spectral norm, we derive

$$\|A_{pj}\| = \sqrt{\|A_{pj}^\top A_{pj}\|} = \sqrt{\|A_{pj} A_{pj}^\top\|} = \|A_{pj}^\top\| \quad \text{and} \quad \|C_{pj}\| \leq \frac{\|A_{pj}\| + \|A_{pj}^\top\|}{2} = \|A_{pj}\|.$$

Thus we will prove that  $\|C_{pj}\| \leq M$  if we show that  $\|A_{pj}\| \leq M$  (for each  $j = 1, 2$ ).

The spectral norm  $\|A_{pj}\|$  is the square root of the largest eigenvalue  $\lambda$  of  $A_{pj}^\top A_{pj}$ . It is attained by the corresponding (real) eigenvector of  $A_{pj}^\top A_{pj}$ , i.e.  $\|A_{pj}\| = \sqrt{\lambda} =$

$\|A_{pj}x\|$  whenever  $(A_{pj}^\top A_{pj})x = \lambda x$  for  $x \in \mathbb{R}^p$  having  $\|x\| = 1$ . The latter implies that

$$\begin{aligned} \|A_{pj}\| &= \sup_{x \in \mathbb{R}^p: \|x\|=1} \|A_{pj}x\| \leq \sup_{x \in \mathbb{R}^p: \|x\|=1} \|A_p x\| \leq \\ &\leq \sup_{y \in \mathbb{C}^p: \|y\|=1} \|A_p y\| = \|A_p\| \leq M. \end{aligned} \quad (22)$$

As a result, to prove (21) we need to verify that

$$\mathbb{E}(|\mathbf{y}_p^\top C_p \mathbf{y}_p - \text{tr}(\Sigma_p C_p)|^2 | \Sigma_p) \leq 8M^2 \text{tr}(\Sigma_p^2) \quad \text{a.s.}$$

for any real symmetric  $p \times p$  matrix  $C_p$  with  $\|C_p\| \leq M$ . Any such matrix can be represented as  $C_p = D_{p1} - D_{p2}$  for real symmetric positive semi-definite  $p \times p$  matrices  $D_{pj}$ ,  $j = 1, 2$ , with  $\|D_{pj}\| \leq \|C_p\|$ . Additionally, by the Cauchy inequality,

$$|\mathbf{y}_p^\top C_p \mathbf{y}_p - \text{tr}(\Sigma_p C_p)|^2 \leq 2 \sum_{j=1}^2 |\mathbf{y}_p^\top D_{pj} \mathbf{y}_p - \text{tr}(\Sigma_p D_{pj})|^2.$$

Thus, to prove (21) we need to verify that

$$\mathbb{E}(|\mathbf{y}_p^\top D_p \mathbf{y}_p - \text{tr}(\Sigma_p D_p)|^2 | \Sigma_p) \leq 2M^2 \text{tr}(\Sigma_p^2) \quad \text{a.s.}$$

for any real symmetric positive semi-definite  $p \times p$  matrix  $D_p$  with  $\|D_p\| \leq M$ .

Let  $B_p = \Sigma_p^{1/2} D_p \Sigma_p^{1/2}$ . Recalling that  $\mathbf{y}_p = \Sigma_p^{1/2} \mathbf{w}_p$ , we have  $\mathbf{y}_p^\top D_p \mathbf{y}_p = \mathbf{w}_p^\top B_p \mathbf{w}_p$ ,

$$\text{tr}(\Sigma_p D_p) = \text{tr}(B_p) \quad \text{and} \quad \text{tr}(B_p^2) = \text{tr}(\Sigma_p D_p \Sigma_p D_p) \leq \|D_p\|^2 \text{tr}(\Sigma_p^2)$$

(see (20)). Writing  $B_p = \sum_{k=1}^p \lambda_{kp} e_{kp} e_{kp}^\top$  for orthonormal vectors  $e_{kp} \in \mathbb{R}^p$ ,  $1 \leq k \leq n$ ,

we get

$$\begin{aligned}\mathbb{E}(|\mathbf{y}_p^\top D_p \mathbf{y}_p - \text{tr}(\Sigma_p D_p)|^2 | \Sigma_p) &= \mathbb{E}(|\mathbf{w}_p^\top B_p \mathbf{w}_p - \text{tr}(B_p)|^2 | \Sigma_p) = \\ &= \text{Var}\left(\sum_{k=1}^p \lambda_{kp} (\mathbf{w}_p^\top e_{kp})^2 | \Sigma_p\right) = \sum_{k=1}^p \lambda_{kp}^2 \text{Var}(\xi) = 2\text{tr}(B_p^2) \leq 2M^2 \text{tr}(\Sigma_p^2)\end{aligned}$$

a.s., since  $\{(\mathbf{w}_p^\top e_{kp})^2\}_{k=1}^p$  are independent random variables distributed as  $\xi \sim \chi_1^2$  (conditionally on  $\Sigma_p$ ).

Assume now that (A1) holds. Let us show that (A2) holds. Take  $A_p = I_p$ . Hence

$$\frac{\mathbf{y}_p^\top \mathbf{y}_p - \text{tr}(\Sigma_p)}{p} = \frac{\mathbf{w}_p^\top \Sigma_p \mathbf{w}_p - \text{tr}(\Sigma_p)}{p} \xrightarrow{p} 0$$

and, in the above definition,  $\lambda_{1p}, \dots, \lambda_{pp}$  are eigenvalues of  $\Sigma_p$ . Suppose also

$$\|\Sigma_p\| = \lambda_{1p} \geq \dots \geq \lambda_{pp} \geq 0.$$

If  $\mathbf{w}_p^*$  is an independent copy of  $\mathbf{w}_p$  and  $\mathbf{w}_p^*$  is also independent of  $\Sigma_p$ , then

$$\frac{\mathbf{w}_p^\top \Sigma_p \mathbf{w}_p - (\mathbf{w}_p^*)^\top \Sigma_p \mathbf{w}_p^*}{p} = \frac{\mathbf{w}_p^\top \Sigma_p \mathbf{w}_p - \text{tr}(\Sigma_p)}{p} - \frac{(\mathbf{w}_p^*)^\top \Sigma_p \mathbf{w}_p^* - \text{tr}(\Sigma_p)}{p} \xrightarrow{p} 0.$$

Therefore,

$$\begin{aligned}\mathbb{E} \exp\{i(\mathbf{w}_p^\top \Sigma_p \mathbf{w}_p - (\mathbf{w}_p^*)^\top \Sigma_p \mathbf{w}_p^*)/p\} &= \\ &= \mathbb{E} \prod_{k=1}^p \exp\{i\lambda_{pk}[(\mathbf{w}_p^\top e_{kp})^2 - ((\mathbf{w}_p^*)^\top e_{kp})^2]/p\} \\ &= \mathbb{E} \prod_{k=1}^p |\varphi(\lambda_{pk}/p)|^2 \rightarrow 1\end{aligned}$$

as  $p \rightarrow \infty$ , where  $\varphi(t) = \mathbb{E} \exp\{it\xi\}$  for  $t \in \mathbb{R}$  and  $\xi \sim \chi_1^2$  as before. Hence,

$$\mathbb{E}|\varphi(\|\Sigma_p\|/p)|^2 = \mathbb{E} \frac{1}{|1 - 2i\|\Sigma_p\|/p|} = \mathbb{E} \frac{1}{(1 + 4\|\Sigma_p\|^2/p^2)^{1/2}} \xrightarrow{p} 1.$$

As a result, we conclude that  $\|\Sigma_p\|/p \xrightarrow{p} 0$  and

$$\begin{aligned} \mathbb{E} \prod_{k=1}^p |\varphi(\lambda_{pk}/p)|^2 &= \mathbb{E} \prod_{k=1}^p \frac{1}{(1 + 4\lambda_{pk}^2/p^2)^{1/2}} \\ &= \mathbb{E} \min \left\{ \exp \left\{ (-2 + \zeta_p) \sum_{k=1}^p \lambda_{pk}^2/p^2 \right\}, 1 \right\} \rightarrow 1 \end{aligned}$$

for some  $\zeta_p$  with  $\zeta_p \xrightarrow{p} 0$ . Thus,  $\text{tr}(\Sigma_p^2)/p^2 \xrightarrow{p} 0$ , i.e. (A2) holds. Q.e.d.

**Proof of Lemma 5.1.** Write  $C = \sum_{k=1}^p \lambda_k e_k e_k^\top$  for orthonormal vectors  $e_k \in \mathbb{R}^p$ ,  $1 \leq k \leq p$ . Then the result follows from inequalities

$$\begin{aligned} |1 + w^\top (C - zI_p)^{-1} w| &\geq \text{Im}(w^\top (C - zI_p)^{-1} w) = \text{Im} \left( \sum_{k=1}^p \frac{(w^\top e_k)^2}{\lambda_k - z} \right) = \\ &= \text{Im}(z) \sum_{k=1}^p \frac{(w^\top e_k)^2}{|\lambda_k - z|^2} \geq \text{Im}(z) \left| \sum_{k=1}^p \frac{(w^\top e_k)^2}{(\lambda_k - z)^2} \right| = \text{Im}(z) |w^\top (C - zI_p)^{-2} w|. \end{aligned}$$

Q.e.d.

**Proof of Lemma 5.2.** The spectral norm of  $A$  is the square root of the largest eigenvalue of  $A^*A$ , where  $A^* = \overline{A}^\top = (C - \bar{z}I_p)^{-1}$ . Write  $z = u + iv$  for  $u \in \mathbb{R}$  and  $v = \text{Im}(z) > 0$ . By definition,

$$A^*A = (C - \bar{z}I_p)^{-1}(C - zI_p)^{-1} = ((C - uI_p)^2 + v^2I_p)^{-1}.$$

Hence, the largest eigenvalue of  $A^*A$  does not exceed  $1/v^2$  and  $\|A\| \leq 1/v$ . Q.e.d.

**Proof of Lemma 5.3.** For any given  $\varepsilon, M > 0$ , set

$$I_0(\varepsilon, M) = \overline{\lim}_{p \rightarrow \infty} \sup_{A_p} \mathbb{P}(|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)| > \varepsilon p), \quad (23)$$

where the supremum is taken over all real symmetric  $p \times p$  matrices  $A_p$  with  $\|A_p\| \leq M$ .

By this definition, there are  $p_k \rightarrow \infty$  and  $A_{p_k}$  with  $\|A_{p_k}\| \leq M$  such that

$$I_0(\varepsilon, M) = \lim_{k \rightarrow \infty} \mathbb{P}(|\mathbf{y}_{p_k}^\top A_{p_k} \mathbf{y}_{p_k} - \text{tr}(\Sigma_{p_k} A_{p_k})| > \varepsilon p_k).$$

Every real symmetric matrix  $A_p$  can be represented as  $A_p = A_{p1} - A_{p2}$  for real symmetric positive semi-definite  $p \times p$  matrices  $A_{pj}$ ,  $j = 1, 2$ , with  $\|A_{pj}\| \leq \|A_p\|$ . Moreover, for any  $\varepsilon > 0$  and  $p \geq 1$ ,

$$\begin{aligned} \mathbb{P}(|\mathbf{y}_p^\top A_p \mathbf{y}_p - \text{tr}(\Sigma_p A_p)| > \varepsilon p) &\leq \mathbb{P}(|\mathbf{y}_p^\top A_{p1} \mathbf{y}_p - \text{tr}(\Sigma_p A_{p1})| > \varepsilon p/2) \\ &\quad + \mathbb{P}(|\mathbf{y}_p^\top A_{p2} \mathbf{y}_p - \text{tr}(\Sigma_p A_{p2})| > \varepsilon p/2). \end{aligned} \quad (24)$$

Hence, it follows from (A1) that  $I_0(\varepsilon, M) = 0$  for any  $\varepsilon, M > 0$ .

If  $A_p$  is any real  $p \times p$  matrix and  $B_p = (A_p^\top + A_p)/2$ , then  $\mathbf{y}_p^\top A_p \mathbf{y}_p = \mathbf{y}_p^\top B_p \mathbf{y}_p$ ,

$$\|A_p\| = \sqrt{\|A_p^\top A_p\|} = \sqrt{\|A_p A_p^\top\|} = \|A_p^\top\| \quad \text{and} \quad \|B_p\| \leq \frac{\|A_p\| + \|A_p^\top\|}{2} = \|A_p\|.$$

In addition,  $\text{tr}(\Sigma_p A_p) = \text{tr}(A_p \Sigma_p) = \text{tr}((A_p \Sigma_p)^\top) = \text{tr}(\Sigma_p A_p^\top) = \text{tr}(\Sigma_p B_p)$ . Thus, if  $I_1(\varepsilon, M)$  is defined as  $I_0(\varepsilon, M)$  in (23) with the supremum taken over all real  $p \times p$  matrices  $A_p$  with  $\|A_p\| \leq M$ , then

$$I_1(\varepsilon, M) = 0 \quad \text{for any } \varepsilon, M > 0. \quad (25)$$



Let now  $A_p = A_{p1} + iA_{p2}$  for real  $p \times p$  matrices  $A_{pj}$ ,  $j = 1, 2$ . It is shown in the proof of Proposition 2.1 (see (22)) that

$$\|A_{pj}\| \leq \|A_p\|, \quad j = 1, 2. \quad (26)$$

Define  $I_2(\varepsilon, M)$  similarly to  $I_0(\varepsilon, M)$  in (23) with the supremum taken over all complex  $p \times p$  matrices  $A_p$  with  $\|A_p\| \leq M$ . Combining (24), (25) and (26) yields  $I_2(\varepsilon, M) = 0$  for any  $\varepsilon, M > 0$ . Q.e.d.

**Proof of Lemma 5.4.** We have

$$\begin{aligned} I &= \frac{z_1}{1+w_1} - \frac{z_2}{1+w_2} = \frac{z_1(1+w_2) - z_2(1+w_1) - z_1w_1 + w_1z_1}{(1+w_1)(1+w_2)} \\ &= \frac{(z_1 - z_2) + z_1(w_2 - w_1) + w_1(z_1 - z_2)}{(1+w_1)(1+w_2)}. \end{aligned}$$

It follows from  $|z_1 - z_2| \leq \gamma$ ,  $|w_1 - w_2| \leq \gamma$  and  $|z_1|/|1+w_1| \leq M$  that

$$|I| \leq \frac{\gamma(1+|z_1|+|w_1|)}{|1+w_1||1+w_2|} \leq \frac{\gamma}{|1+w_2||1+w_1|} + \frac{\gamma M}{|1+w_2|} + \frac{\gamma}{|1+w_2|} \frac{|w_1|}{|1+w_1|}.$$

In addition, we have  $|1+w_2| \geq \delta$ ,

$$|1+w_1| = |1+w_2 + (w_1 - w_2)| \geq \delta - \gamma \geq \delta/2,$$

$$\frac{|w_1|}{|1+w_1|} = \frac{2}{|1+w_1|} I(|w_1| \leq 2) + \frac{|w_1|}{|w_1| - 1} I(|w_1| > 2) \leq \begin{cases} 4/\delta, & |w_1| \leq 2, \\ 2, & |w_1| > 2. \end{cases}$$

Finally, we conclude that  $|I| \leq \gamma(M/\delta + 4/\min\{\delta^2, 2\delta\} + 2/\delta^2)$ . Q.e.d.

**Proof of Lemma 5.5.** Write  $z = u + iv$  for  $u \in \mathbb{R}$  and  $v > 0$ . We need to prove

inequality

$$|z||1 + \operatorname{tr}(\Sigma(C - zI_p)^{-1})| = |z + \operatorname{tr}(\Sigma(C/z - I_p)^{-1})| \geq v.$$

Since

$$|z + \operatorname{tr}(\Sigma(C/z - I_p)^{-1})| \geq \operatorname{Im}(z + \operatorname{tr}(\Sigma(C/z - I_p)^{-1})) = v + \operatorname{Im}(\operatorname{tr}(\Sigma(C/z - I_p)^{-1})),$$

we only need to check that

$$\operatorname{Im}(\operatorname{tr}(\Sigma(C/z - I_p)^{-1})) \geq 0.$$

Denote

$$B = \left( \frac{u}{|z|^2} C - I_p \right)^2 + \frac{v^2}{|z|^4} C^2. \quad (27)$$

Note that  $B$  is an invertible symmetric positive definite matrix, since

$$\begin{aligned} B &= (C/z - I_p)(C/z - I_p)^* = (C/z - I_p)^*(C/z - I_p) = \\ &= (C/\bar{z} - I_p)(C/z - I_p) = \frac{1}{|z|^2} C^2 - \frac{2u}{|z|^2} C + I_p \end{aligned}$$

and  $C/z - I_p = (C - zI_p)/z$  is invertible, where  $A^* = \overline{A}^\top$  is the conjugate transpose of a matrix  $A$ . Additionally,

$$(C/z - I_p)^{-1} = (C/\bar{z} - I_p)B^{-1} = \left( \frac{u}{|z|^2} C - I_p + \frac{iv}{|z|^2} C \right) B^{-1}.$$

Therefore,

$$\operatorname{Im}(\operatorname{tr}(\Sigma(C/z - I_p)^{-1})) = \frac{v}{|z|^2} \operatorname{Im}(\operatorname{tr}(\Sigma C B^{-1})).$$

Let  $C^{1/2}$  and  $\Sigma^{1/2}$  be the principal square roots of  $C$  and  $\Sigma$ . Then, by the definition of  $B$ , matrices  $C^{1/2}$  and  $B^{-1}$  commute,  $CB^{-1} = C^{1/2}B^{-1}C^{1/2}$  and

$$\text{tr}(\Sigma CB^{-1}) = \text{tr}(\Sigma C^{1/2} B^{-1} C^{1/2}) = \text{tr}(\Sigma^{1/2} C^{1/2} B^{-1} C^{1/2} \Sigma^{1/2}).$$

As it is shown above,  $B$  is symmetric positive definite. Hence,  $B^{-1}$  is symmetric positive definite and  $QB^{-1}Q^\top$  is symmetric positive semi-definite for any  $p \times p$  matrix  $Q$ . Taking  $Q = \Sigma^{1/2}C^{1/2}$ , we see that

$$\text{tr}(\Sigma CB^{-1}) = \text{tr}(QB^{-1}Q^\top) \geq 0.$$

This proves the desired bound. Q.e.d.

**Proof of Lemma 5.6.** It is shown in the proof of Lemma 5.2 that, for all  $\varepsilon, M > 0$  and each  $k = 1, \dots, n$ ,

$$\begin{aligned} \sup_{A_p} \mathbb{P}(|\mathbf{y}_{pk}^\top A_p \mathbf{y}_{pk} - \text{tr}(\Sigma_{pk} A_p)| > \varepsilon p) &\leq 2 \sup_{B_p} \mathbb{P}(|\mathbf{y}_{pk}^\top B_p \mathbf{y}_{pk} - \text{tr}(\Sigma_{pk} B_p)| > \varepsilon p/2) \\ &\leq 2 \sup_{C_p} \mathbb{P}(|\mathbf{y}_{pk}^\top C_p \mathbf{y}_{pk} - \text{tr}(\Sigma_{pk} C_p)| > \varepsilon p/2) \\ &\leq 4 \sup_{D_p} \mathbb{P}(|\mathbf{y}_{pk}^\top D_p \mathbf{y}_{pk} - \text{tr}(\Sigma_{pk} D_p)| > \varepsilon p/4), \end{aligned}$$

where  $A_p$  is a complex  $p \times p$  matrix with  $\|A_p\| \leq M$ ,  $B_p$  is a real  $p \times p$  matrix with  $\|B_p\| \leq M$ ,  $C_p$  is a real symmetric  $p \times p$  matrix with  $\|C_p\| \leq M$ , and  $D_p$  is a real symmetric positive semi-definite  $p \times p$  matrix with  $\|D_p\| \leq M$ . Thus the result follows from (A3). Q.e.d.

**Proof of Lemma 5.7.** In what follows, we denote the principal square root of  $Q$  by  $Q^{1/2}$  for any real symmetric positive semi-definite  $p \times p$  matrix  $Q$ . For each  $z = u + iv$

with  $u \in \mathbb{R}$  and  $v = \text{Im}(z) > 0$ ,

$$\|(I_q + U^\top(C - zI_p)^{-1}U)^{-1}\| = |z| \|(zI_q + U^\top(C/z - I_p)^{-1}U)^{-1}\|.$$

Arguing as in the proof of Lemma 5.5, we derive

$$(C/z - I_p)^{-1} = \left(\frac{u}{|z|^2}C - I_p\right)B^{-1} + \frac{iv}{|z|^2}CB^{-1} = A_1 + iA_2,$$

where  $B$  is given in (27),  $A_1$  and  $A_2$  are real symmetric commuting  $p \times p$  matrices defined by

$$A_1 = \frac{u}{|z|^2}C^{1/2}B^{-1}C^{1/2} - B^{-1}, \quad A_2 = \frac{v}{|z|^2}C^{1/2}B^{-1}C^{1/2}.$$

In addition,  $A_2$  is symmetric positive semi-definite. We have

$$\begin{aligned} \|(zI_q + U^\top(C/z - I_p)^{-1}U)^{-1}\| &= \|((uI_q + U^\top A_1 U) + i(vI_q + U^\top A_2 U))^{-1}\| \\ &\leq \|A_3^{-1/2}(A_3^{-1/2}(uI_q + U^\top A_1 U)A_3^{-1/2} + iI_q)^{-1}A_3^{-1/2}\| \\ &\leq \|A_3^{-1/2}\|^2 \|(A_3^{-1/2}(uI_q + U^\top A_1 U)A_3^{-1/2} + iI_q)^{-1}\|, \end{aligned}$$

where  $A_3 = vI_p + U^\top A_2 U$ . Since  $U^\top A_2 U$  is a symmetric positive semi-definite matrix, the spectral norm of  $A_3^{-1/2}$  does not exceed  $1/\sqrt{v}$ . The spectral norm of

$$A_4 = (A_3^{-1/2}(uI_q + U^\top A_1 U)A_3^{-1/2} + iI_q)^{-1}$$

is the largest eigenvalue of

$$A_4^* A_4 = ((A_3^{-1/2}(uI_q + U^\top A_1 U)A_3^{-1/2})^2 + I_q)^{-1}.$$

Obviously, it is not greater than 1. Finally, we conclude that

$$\|(I_q + U^\top(C - zI_p)^{-1}U)^{-1}\| \leq \frac{|z|}{v}.$$

Q.e.d.

**Proof of Lemma 5.8.** As above, we denote the principal square root of  $Q$  by  $Q^{1/2}$  for any real symmetric positive semi-definite  $p \times p$  matrix  $Q$ . Let also  $R^* = \overline{R}^\top$  be the conjugate transpose of a matrix (or a vector)  $R$ .

Write  $z = u + iv$  for  $u \in \mathbb{R}$  and  $v > 0$  and set

$$A_0 = (C - zI_p)^{-1}U(I_q + U^\top(C - zI_p)^{-1}U)^{-1}.$$

We need to prove that

$$|w^\top A_0^\top A_0 w| \leq \frac{v + |z|}{v^2} \|w\|^2.$$

By the Cauchy-Schwartz inequality,

$$|w^\top A_0^\top A_0 w| = |(\overline{A_0 w}, A_0 w)| \leq \|\overline{A_0 w}\| \|A_0 w\| = \|A_0 w\|^2 \leq \|A_0\|^2 \|w\|^2.$$

By Theorem A.6 in [1],

$$\begin{aligned} (C - zI_p)^{-1} &= (C - uI_p - ivI_q)^{-1} = (C - uI_p + ivI_p)B^{-1} = \\ &= CB^{-1} - \bar{z}B^{-1} = C^{1/2}B^{-1}C^{1/2} - \bar{z}B^{-1}, \end{aligned}$$

$$[(C - zI_p)^{-1}]^*(C - zI_p)^{-1} = (C - \bar{z}I_p)^{-1}(C - zI_p)^{-1} = B^{-1},$$

where

$$B = (C - uI_p)^2 + v^2 I_p \quad (28)$$

is a real symmetric positive definite matrix commuting with  $C^{1/2}$  (its inverse  $B^{-1}$  has the same properties). We also have

$$\begin{aligned} (I_q + U^\top (C - zI_p)^{-1} U)^{-1} &= (A_1 - \bar{z} U^\top B^{-1} U)^{-1} \\ &= A_1^{-1/2} (I_q - \bar{z} A_1^{-1/2} U^\top B^{-1} U A_1^{-1/2})^{-1} A_1^{-1/2} \\ &= A_1^{-1/2} (I_q - \bar{z} V^\top V)^{-1} A_1^{-1/2} \end{aligned}$$

and  $A_0 = A_2 A_1^{-1/2}$ , where  $V = B^{-1/2} U A_1^{-1/2}$ ,

$$\begin{aligned} A_1 &= I_q + U^\top C^{1/2} B^{-1} C^{1/2} U, \\ A_2 &= (C - zI_p)^{-1} B^{1/2} V (I_p - \bar{z} V^\top V)^{-1}. \end{aligned} \quad (29)$$

The matrix  $Q^\top B^{-1} Q$  is real symmetric positive semi-definite for any  $p \times p$  matrix  $Q$ . Therefore, taking  $Q = C^{1/2} U$  yields

$$\|A_1^{-1/2}\| = \|(I_q + Q^\top B^{-1} Q)^{-1/2}\| = \|(I_q + Q^\top B^{-1} Q)^{-1}\|^{1/2} \leq 1$$

and  $\|A_0\| \leq \|A_2\| \|A_1^{-1/2}\| \leq \|A_2\|$ . The spectral norm  $\|A_2\|$  is equal to the square root of the spectral norm of the matrix

$$\begin{aligned} A_2^* A_2 &= [(I_p - \bar{z} V^\top V)^{-1}]^* V^\top B^{1/2} [(C - zI_p)^{-1}]^* (C - zI_p)^{-1} B^{1/2} V (I_p - \bar{z} V^\top V)^{-1} \\ &= (I_p - z V^\top V)^{-1} V^\top V (I_p - \bar{z} V^\top V)^{-1}. \end{aligned}$$

Additionally,

$$\begin{aligned}\|A_2^* A_2\| &\leq \frac{1}{|z|^2} \|(V^\top V - \bar{z}I_p/|z|^2)^{-1} V^\top V (V^\top V - zI_p/|z|^2)^{-1}\| \\ &\leq \frac{1}{|z|^2} \|(V^\top V - \bar{z}I_p/|z|^2)^{-1}\| \|V^\top V (V^\top V - zI_p/|z|^2)^{-1}\|.\end{aligned}$$

By the triangle inequality,

$$\begin{aligned}\|V^\top V (V^\top V - zI_p/|z|^2)^{-1}\| &= \|I_p + z|z|^{-2} (V^\top V - zI_p/|z|^2)^{-1}\| \\ &\leq 1 + \frac{1}{|z|} \|(V^\top V - zI_p/|z|^2)^{-1}\|\end{aligned}$$

Matrices  $A_3 = (V^\top V - zI_p/|z|^2)^{-1}$  and  $A_3^* = (V^\top V - \bar{z}I_p/|z|^2)^{-1}$  have identical spectral norms equal to the square root of the largest eigenvalue of the matrix

$$A_3^* A_3 = A_3 A_3^* = ((V^\top V - uI_p/|z|^2)^2 + v^2 I_p/|z|^4)^{-1}.$$

Obviously, this eigenvalue does not exceed  $|z|^4/v^2$ . Combining the above estimates yields

$$|w^\top A_0^\top A_0 w| \leq \|A_0\|^2 \|w\|^2 \leq \frac{1}{|z|^2} \frac{|z|^2}{v} \left(1 + \frac{1}{|z|} \frac{|z|^2}{v}\right) \|w\|^2 = \frac{v + |z|}{v^2} \|w\|^2.$$

**Proof of Lemma 5.9.** Let  $z = u + iv$  for  $u \in \mathbb{R}$ ,  $v = \text{Im}(z) > 0$  and  $C_z = C - zI_p$ .

It follows from Lemma 5.7 that

$$A_z = I_q + U^\top C_z^{-1} U$$

is non-degenerate. By the Sherman-Morrison-Woodbury formula,

$$(C_z + UU^\top)^{-1} = C_z^{-1} - C_z^{-1}UA_z^{-1}U^\top C_z^{-1}. \quad (30)$$

The Cauchy-Schwartz inequality, (30) and Lemma 5.7 imply that

$$\begin{aligned} |y^\top (C_z + UU^\top)^{-1}y - y^\top C_z^{-1}y| &\leq (y^\top C_z^{-1}U)A_z^{-1}(U^\top C_z^{-1}y) \\ &\leq \|\overline{U^\top C_z^{-1}y}\| \|A_z^{-1}(U^\top C_z^{-1}y)\| \\ &\leq \frac{|z|}{v} \|U^\top C_z^{-1}y\|^2. \end{aligned}$$

It also follows from (30) that

$$\begin{aligned} y^\top (C_z + UU^\top)^{-2}y &= y^\top (C_z^{-1} - C_z^{-1}UA_z^{-1}U^\top C_z^{-1})^2y \\ &= y^\top C_z^{-2}y + (y^\top C_z^{-1}U)A_z^{-1}U^\top C_z^{-2}UA_z^{-1}(U^\top C_z^{-1}y) - \\ &\quad - (y^\top C_z^{-1}U)A_z^{-1}(U^\top C_z^{-2}y) - (y^\top C_z^{-2}U)A_z^{-1}(U^\top C_z^{-1}y). \end{aligned}$$

Applying the Cauchy-Schwartz inequality, this identity, Lemma 5.7 and Lemma 5.8, we get

$$\begin{aligned} |y^\top (C_z + UU^\top)^{-2}y - y^\top C_z^{-2}y| &\leq \frac{2|z|}{v} \|U^\top C_z^{-1}y\| \|U^\top C_z^{-2}y\| + \frac{v + |z|}{v^2} \|U^\top C_z^{-1}y\|^2 \\ &\leq \frac{|z|}{v} \|U^\top C_z^{-1}y\|^2 + \frac{|z|}{v} \|U^\top C_z^{-2}y\|^2 + \frac{v + |z|}{v^2} \|U^\top C_z^{-1}y\|^2 \\ &\leq \frac{(|z| + 1)^2}{v^2} \sum_{j=1}^2 \|U^\top C_z^{-j}y\|^2. \end{aligned}$$

Gathering together the above estimates, we finish the proof. Q.e.d.

**Proof of Lemma 5.10.** Let  $z = u + iv$  for  $u \in \mathbb{R}$  and  $v = \text{Im}(z) > 0$ . Denote further by  $Q^*$  the conjugate transpose of a matrix  $Q$ , i.e.  $Q^* = \overline{Q}^\top$ .



Recall that  $\text{tr}(Q_1 Q_2) = \text{tr}(Q_2 Q_1)$ ,  $(Q_1 Q_2)^* = Q_2^* Q_1^*$ ,  $(Q^{-1})^* = (Q^*)^{-1}$ ,  $(Q^*)^* = Q$ ,

$$\|Q^*\| = \sqrt{\|Q Q^*\|} = \sqrt{\|Q^* Q\|} = \|Q\|,$$

$$|\text{tr}(Q)|^2 = |\text{tr}(I_q^* Q)|^2 \leq \text{tr}(I_q^* I_q) \text{tr}(Q^* Q) = q \text{tr}(Q^* Q) \leq q^2 \|Q\|^2 \quad (31)$$

for any complex  $q \times q$  matrices  $Q, Q_1, Q_2$ . Hence, taking  $Q = AU(I_q + U^\top AU)^{-1}U^\top A$  we arrive at the bound

$$|\text{tr}(U^\top A^2 U(I_q + U^\top AU)^{-1})|^2 = |\text{tr}(Q)|^2 \leq q \text{tr}(Q^* Q),$$

where

$$\begin{aligned} \text{tr}(Q^* Q) &= \text{tr}(A^* U(I_q + U^\top A^* U)^{-1} U^\top A^* A U(I_q + U^\top AU)^{-1} U^\top A) \\ &= \text{tr}(A_1^{-1/2} U^\top A A^* U(I_q + U^\top A^* U)^{-1} A_1^{1/2} A_1^{-1/2} U^\top A^* A U(I_q + U^\top AU)^{-1} A_1^{1/2}) \end{aligned}$$

and  $A_1$  is given in (29). Thus,  $\text{tr}(Q^* Q)$  is bounded from above by

$$\begin{aligned} I &= q \|A_1^{-1/2} U^\top A A^* U(I_q + U^\top A^* U)^{-1} A_1^{1/2}\| \|A_1^{-1/2} U^\top A^* A U(I_q + U^\top AU)^{-1} A_1^{1/2}\| \\ &= q \|A_1^{1/2} (I_q + U^\top AU)^{-1} U^\top A A^* U A_1^{-1/2}\| \|A_1^{-1/2} U^\top A^* A U(I_q + U^\top AU)^{-1} A_1^{1/2}\| \end{aligned}$$

Since  $A = (C - zI_p)^{-1}$ ,  $A^* = (C - \bar{z}I_p)^{-1}$  and  $A^* A = A A^* = B^{-1}$  for  $B$  defined in (28), we can proceed further as in the proof of Lemma 5.8 using the same notation. Namely,

$$A_1^{-1/2} U^\top A^* A U(I_q + U^\top AU)^{-1} A_1^{1/2} = V^\top V(I_q - \bar{z}V^\top V)^{-1},$$

$$A_1^{1/2} (I_q + U^\top AU)^{-1} U^\top A A^* U A_1^{-1/2} = (I_q - \bar{z}V^\top V)^{-1} V^\top V.$$

By the same arguments as in the proof of Lemma 5.8,

$$\|V^\top V(I_q - \bar{z}V^\top V)^{-1}\| \leq \frac{1}{|z|} \left(1 + \frac{|z|}{v}\right),$$

$$\|(I_q - \bar{z}V^\top V)^{-1}V^\top V\| \leq \frac{1}{|z|} \left(1 + \frac{|z|}{v}\right).$$

Finally we conclude that

$$|\operatorname{tr}(U^\top A^2 U(I_q + U^\top A U)^{-1})| \leq \frac{q(|z| + v)}{|z|v}.$$

Q.e.d.

**Proof of Lemma 5.11.** Let  $z = u + iv$  for  $u \in \mathbb{R}$  and  $v = \operatorname{Im}(z) > 0$ ,

$$\Delta = V^\top A A^* V - U^\top A A^* U \quad \text{and} \quad \Delta_j = V^\top A^j V - U^\top A^j U, \quad j = 1, 2.$$

Denote further by  $Q^*$  the conjugate transpose of a matrix  $Q$ , i.e.  $Q^* = \overline{Q}^\top$ .

We have

$$\begin{aligned} & |\operatorname{tr}(V^\top A^2 V(I_q + V^\top A V)^{-1}) - \operatorname{tr}(U^\top A^2 U(I_q + U^\top A U)^{-1})| \leq \\ & \leq |\operatorname{tr}(V^\top A^2 V(I_q + V^\top A V)^{-1}) - \operatorname{tr}(U^\top A^2 U(I_q + V^\top A V)^{-1})| \\ & \quad + |\operatorname{tr}(U^\top A^2 U(I_q + V^\top A V)^{-1}) - \operatorname{tr}(U^\top A^2 U(I_q + U^\top A U)^{-1})| \\ & \leq |\operatorname{tr}(\Delta_2(I_q + V^\top A V)^{-1})| + |\operatorname{tr}(U^\top A^2 U(I_q + U^\top A U)^{-1} \Delta_1(I_q + V^\top A V)^{-1})| \end{aligned}$$

By Lemma 5.7 and (31),

$$\begin{aligned}
|\operatorname{tr}(\Delta_2(I_q + V^\top AV)^{-1})| &\leq q \|\Delta_2(I_q + V^\top AV)^{-1}\| \\
&\leq q \|\Delta_2\| \|(I_q + V^\top AV)^{-1}\| \\
&\leq \frac{q|z|}{v} \|\Delta_2\|.
\end{aligned}$$

Taking  $Q = AU(I_q + U^\top AU)^{-1}\Delta_1(I_q + V^\top AV)^{-1}U^\top A$  we infer that

$$|\operatorname{tr}(U^\top A^2 U(I_q + U^\top AU)^{-1})\Delta_1(I_q + V^\top AV)^{-1})| = |\operatorname{tr}(Q)| \leq q \|Q\|$$

and

$$\|Q\| \leq \|AU(I_q + U^\top AU)^{-1}\| \|\Delta_1\| \|(I_q + V^\top AV)^{-1}U^\top A\|.$$

It is shown in the proof of Lemma 5.8 (where  $A_0 = AU(I_q + U^\top AU)^{-1}$ ) that

$$\|AU(I_q + U^\top AU)^{-1}\| \leq \frac{\sqrt{v + |z|}}{v}. \quad (32)$$

In addition, since  $\|R\|^2 = \|R^*R\| = \|RR^*\|$  for any complex  $q \times p$  matrix  $R$ , we infer that

$$\begin{aligned}
\|(I_q + V^\top AV)^{-1}U^\top A\|^2 &= \|(I_q + V^\top AV)^{-1}U^\top AA^*U[(I_q + V^\top AV)^{-1}]^*\| \\
&\leq \|(I_q + V^\top AV)^{-1}\Delta[(I_q + V^\top AV)^{-1}]^*\| \\
&\quad + \|(I_q + V^\top AV)^{-1}V^\top AA^*V[(I_q + V^\top AV)^{-1}]^*\| \\
&\leq \|(I_q + V^\top AV)^{-1}\|^2 \|\Delta\| + \|(I_q + V^\top AV)^{-1}V^\top A\|^2.
\end{aligned}$$

As in the proof of Lemma 5.8, one can show that

$$\|(I_q + V^\top AV)^{-1}V^\top A\| \leq \frac{\sqrt{v + |z|}}{v}.$$

By Lemma 5.7,

$$\|(I_q + V^\top AV)^{-1}\| \leq \frac{|z|}{v}.$$

Combining the above estimates yields

$$\begin{aligned} |\text{tr}(Q)| &\leq q \frac{\sqrt{v + |z|}}{v} \|\Delta_1\| \left( \frac{|z|^2}{v^2} \|\Delta\| + \frac{v + |z|}{v^2} \right)^{1/2} \\ &\leq \frac{q|z|\sqrt{v + |z|}}{v^2} \|\Delta_1\| \sqrt{\|\Delta\|} + \frac{q(v + |z|)}{v^2} \|\Delta_1\| \\ &\leq \frac{2q(|z| + 1)^{3/2}}{v^2} (\|\Delta_1\| \sqrt{\|\Delta\|} + \|\Delta_1\|). \end{aligned}$$

The latter gives the desired bound

$$\begin{aligned} |\text{tr}(V^\top A^2 V (I_q + V^\top AV)^{-1}) - \text{tr}(U^\top A^2 U (I_q + U^\top AU)^{-1})| &\leq \\ &\leq \frac{2q(|z| + 1)^{3/2}}{v^2} (\|\Delta_1\| \sqrt{\|\Delta\|} + \|\Delta_1\| + \|\Delta_2\|). \end{aligned}$$

Q.e.d.

## References

- [1] Bai, Z., and Silverstein, J.: *Spectral analysis of large dimensional random matrices*. Second edition. New York: Springer, 2010.
- [2] Bai, Z., and Zhou, W.: Large sample covariance matrices without independence structures in columns. *Stat. Sinica*, **18**, (2008), 425-442.

- [3] Banerjee, S., Bose, A., and Sen, S.: Heteroscedastic Wigner matrices. Preprint.
- [4] Banna, M., and Merlevéde, F.: Limiting spectral distribution of large sample covariance matrices associated to a class of stationary processes. *J. of Theor. Probab.*, (2013), 1–39.
- [5] Banna, M., Merlevéde, F., and Peligrad, M.: On the limiting spectral distribution for a large class of symmetric random matrices with correlated entries. ARXIV:1312.0037v2.
- [6] Gaposhkin, V.F.: On the convergence of series of weakly multiplicative systems of functions. *Math. USSR-Sb.*, **18**, (1972), 361–371.
- [7] Girko, V., and Gupta, A.K.: Asymptotic behavior of spectral function of empirical covariance matrices. *Random Oper. and Stoch. Eqs.*, **2(1)**, (1994), 44–60.
- [8] Götze, F., Naumov, A.A., and Tikhomirov, A.N.: *Limit theorems for two classes of random matrices with dependent entries*, *Teor. Veroyatnost. i Primenen.*, **59(1)**, (2014), 61-80. [In Russian]
- [9] Hall, P., and Heyde, C.C.: *Martingale Limit Theory and its Application*. Academic Press, Boston, 1980.
- [10] Horn, A.R., and Johnson, C.R.: *Matrix analysis*. Cambridge University Press, Cambridge, 1990.
- [11] Hui, J., and Pan, G.M.: Limiting spectral distribution for large sample covariance matrices with  $m$ -dependent elements. *Commun. Stat. – Theory Methods*, **39**, (2010), 935–941.

- [12] El Karoui, N.: Spectrum estimation for large dimensional covariance matrices using random matrix theory, *Annals of Statistics*, **36**, (2010), 2757–2790.
- [13] Marcenko, V.A., and Pastur, L.A.: Distribution of eigenvalues in certain sets of random matrices, *Mat. Sb. (N.S.)*, **72**, (1967), 507–536.
- [14] Marshall, A.W., Olkin, I., and Arnold, B.C.: *Inequalities: theory of majorization and Its applications*. Springer, 2010.
- [15] Merlevede, F., and Peligrad, M.: On the empirical spectral distribution for matrices with long memory and independent rows. arXiv:1406.1216.
- [16] Paul, D., and Aue, A.: Random matrix theory in statistics: a review. *Journal of Statistical Planning and Inference*, **150**, (2014), 1–29.
- [17] Pfaffel, O., and Schlemm, E.: Eigenvalue distribution of large sample covariance matrices of linear processes. *Probab. Math. Statist.*, **31**, (2011), 313–329.
- [18] Pajor, A., and Pastur L.: On the limiting empirical measure of eigenvalues of the sum of rank one matrices with log-concave distribution. *Studia Math.*, **195**, (2009), 11–29.
- [19] Pastur, L., and Shcherbina, M.: *Eigenvalue distribution of large random matrices*. Mathematical Surveys and Monographs, **171**. American Mathematical Society, Providence, RI, 2011.
- [20] Tao T.: *Topics in random matrix theory*. Graduate Studies in Mathematics, **132**. American Mathematical Society, Providence, RI, 2012.

- [21] Yin, Y.Q., and Krishnaiah, P.R.: Limit theorems for the eigenvalues of product of large-dimensional random matrices when the underlying distribution is isotropic. *Teor. Veroyatnost. i Primenen.*, **31**, (1986), 394–398.
- [22] Yao, J.: A note on a Marćenko-Pastur type theorem for time series. *Statist. Probab. Lett.*, **82**, (2012), 22–28.